AARMS 5910: q-SERIES IN ANALYSIS

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Abstract. We will develop the theory of combinatorial and analytic identities, summation theorems, and related topics through analytic and combinatorial techniques. The combinatorics involves counting subspaces and mapping vector spaces over finite fields, and partitioning theoretic identities of number theory. The Möbius function on partially ordered sets will also be mentioned.

We will pay special attention to identities like the Rogers-Ramanujan identities and their various generalizations in some detail. A central piece of the analytic development is the Askey–Wilson integral and its generalizations.

Over all the course will be a bridge between analysis and discrete mathematics through the use of combinatorial and analytic tools. The treatment we propose is very conceptual and is a major improvement over the earlier approaches.

The classical approach to q-series is available in [AAR99] and [GR04]. One classic reference on partitions and number theory is [And98].

The lectures will be based on the lecture notes [IS]. A copy of these notes will be made available to the students in the class.

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Date: 22 July 2019.

1. Introduction to q-series

We shall define a few notations that will be used throughout this course.

Definition 1.1. The *q-Pochhammer symbol* or the *q-shifted factorial* is defined by

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

More generally, we shall define

$$(a_1,\ldots,a_k;q)_n := \prod_{j=1}^{k}(a_j;q)_n$$

and

$$\frac{(q;q)_n}{(1-q)^n} = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}.$$

Remark. Observe that

$$\begin{aligned}\lim_{q\to1^-}\frac{(q;q)_n}{(1-q)^n} &= \lim_{q\to1^-}\frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}\\ &= \lim_{q\to1^-}\prod_{k=1}^{n}\frac{1-q^k}{1-q} = \lim_{q\to1^-}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1}) = n!.\end{aligned}$$

We also have the q-analogue of the binomial coefficients and the gamma function.

Definition 1.2. The *q-gamma function* is defined by

$$\Gamma_q(x) := \frac{(1-q)^{1-x}(q;q)_\infty}{(q^x;q)_\infty} = (1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}},$$

provided that $|q|<1$. The *q-binomial coefficient* is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Remark. We have

$$\begin{aligned}\frac{\Gamma_q(x+1)}{\Gamma_q(x)} &= \frac{(1-q)^{-x}}{(1-q)^{1-x}}\cdot\frac{(q^x;q)_\infty}{(q^{x+1};q)_\infty}\\ &= \frac{(1-q^x)(1-q^{x+1})\cdots}{(1-q)(1-q^{x+1})\cdots} = \frac{1-q^x}{1-q}.\end{aligned}$$

2. q-Taylor series

Let $f(z)$ be a function; recall that the Taylor series centred at $x = a$ is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

We want some analogue of the q-series. To do so we need a reliable way to compute the coefficients (the counterparts of a_n) and a reliable basis (i.e., the counterparts of $(x-a), (x-a)^2, \dots$). Recall that the standard basis of any Taylor series centred at $x = a$ satisfies a few properties:

(1) If we let $z = a$, then only the constant term remains, namely $f(a)$.
(2) The term-by-term differentiation lowers the degree of each term by 1.

Our goal of this section is to define the q-analogue of the classic differential operator, find a desirable basis with respect to this q-differential operator (known as the Askey–Wilson divided difference operator, which will be defined in the next section), and justify that a basis we chose (known as the Askey–Wilson basis) does have the properties we want it to hold.

2.1. q-difference operators

We start with the difference operator that F. H. Jackson came up with in 1906.

Definition 2.1. D_q is the *(Jackson) q-difference operator* which is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}.$$

But this is not the type of operator we will mainly use.

To motivate the new definition, think of the following problem. Suppose we want to write $\cos n\theta$ in terms of $\cos\theta$ – in other words, what is f such that $\cos n\theta = f(\cos\theta)$? Note that $\cos 2\theta = 2\cos^2\theta - 1$, so in this case we have $f(x) = 2x^2 - 1$. As for the bigger n's, we can use de Moivre's theorem to find some polynomial T_n such that $T_n(\cos\theta) = \cos n\theta$. Similarly, one can find another polynomial $U_n(x)$ that satisfies

$$\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta).$$

This gives rise to the Chebyshev polynomials.

Definition 2.2. T_n is the *Chebyshev polynomial of the first kind*; U_n is called the *Chebyshev polynomial of the second kind*.

What are the relationships between the two kinds of Chebyshev polynomials? Differentiating $T_n(x)$ where $x = \cos\theta$ yields

$$\frac{dT_n(x)}{dx} = \frac{-n\sin(n\theta)}{-\sin\theta} = nU_{n-1}(x).$$

Suppose $z = e^{i\theta}$. Then since $x = \cos\theta$, we have $x = \frac{1}{2}(z + \frac{1}{z})$, and by a change of variables, there is some $\breve{f}$ such that $f(x) = \breve{f}(z)$. Now we are ready to define another type of difference operators.

Definition 2.3. Define $\mathcal{D}_q$ (called the *Askey–Wilson divided difference operator*) by

$$\mathcal{D}_q f(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{\breve{\mathrm{id}}(q^{1/2}z) - \breve{\mathrm{id}}(q^{-1/2}z)},$$

where $\mathrm{id}(x) = x$ and $\breve{\mathrm{id}}(z) = \frac{1}{2}(z + \frac{1}{z})$. Note that we can simplify the denominator:

$$\begin{aligned}\breve{\mathrm{id}}(q^{1/2}z) - \breve{\mathrm{id}}(q^{-1/2}z) &= \frac{1}{2}(q^{1/2}z + q^{-1/2}z) - \frac{1}{2}(q^{-1/2}z + q^{1/2}z) \\ &= \frac{1}{2}(q^{1/2} - q^{-1/2})\left(z - \frac{1}{z}\right).\end{aligned}$$

Thus we have

$$\mathcal{D}_q f(z) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{\frac{1}{2}(q^{1/2} - q^{-1/2})(z - \frac{1}{z})}.$$

Let's go back to the $T_n(x)$. Indeed, we have $T_n(x) = \breve{T}_n(z) = (z^n + z^{-n})/2$, so

$$\begin{aligned}\mathcal{D}_q T_n(x) &= \frac{\frac{1}{2}(z^n - z^{-n})(q^{n/2} - q^{-n/2})}{\frac{1}{2}(z - z^{-1})(q^{1/2} - q^{-1/2})} \\ &= \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}\left(\frac{i(z^n - z^{-n})/(2i)}{i(z - z^{-1})/(2i)}\right) \\ &= \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}\left(\frac{i(\sin n\theta)}{i(\sin\theta)}\right) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}U_{n-1}(x).\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\lim_{q\to 1^-} \mathcal{D}_q T_n(x) &= \lim_{q\to 1^-} \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}U_{n-1}(x) \\ &= \lim_{q\to 1^-} \frac{(q^{1/2} - q^{-1/2})(q^{(n-1)/2} + q^{(n-2)/2}q^{-1/2} + \cdots + q^{-(n-1)/2})}{q^{1/2} - q^{-1/2}}U_{n-1}(x) \\ &= \lim_{q\to 1^-} (q^{(n-1)/2} + q^{(n-2)/2}q^{-1/2} + \cdots + q^{-(n-1)/2})U_{n-1}(x) \\ &= nU_{n-1}(x) = \frac{d}{dx}T_n(x).\end{aligned}$$

But since $\{T_0(x), T_1(x), \dots\}$ forms a basis of the vector space of polynomials (note that $\deg T_n(x) = n$), we have

$$\lim_{q\to 1^-} \mathcal{D}_q f(x) = \frac{d}{dx}f(x)$$

for any polynomial $f(x)$.

2.2. **Askey–Wilson basis**

Now we shall introduce a new basis (the q-counterpart of $(x-a)^n$) for the q-Taylor polynomials.

Definition 2.4. For $x = \cos\theta$ and $z = e^{i\theta}$, define

$$\phi_n(x;a) = (az, az^{-1}; q)_n = \prod_{k=0}^{n-1}(1 - q^k az)\left(1 - \frac{q^k a}{z}\right) = \prod_{k=0}^{n-1}(1 - 2xaq^k + a^2 q^{2k}),$$

with the last equality following from the fact that $z+z^{-1}=2x$. (Just expand $(1-q^kaz)(1-q^kaz^{-1})$, and use $z+z^{-1}=2x$.) The $\phi_n(x;a)$ are called the *Askey–Wilson basis.*

Upon applying the Askey-Wilson operator to this polynomial, we see

$$\begin{aligned}
\mathcal{D}_q\phi_n(x;a) &= \frac{(aq^{\frac{1}{2}}z, aq^{-\frac{1}{2}}z^{-1};q)_n - (aq^{-\frac{1}{2}}z, aq^{\frac{1}{2}}z^{-1};q)_n}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})\left(z-\frac{1}{z}\right)\frac{1}{2}} \\
&= \frac{(aq^{\frac{1}{2}}z, aq^{\frac{1}{2}}z^{-1};q)_{n-1}[(1-azq^{n-\frac{1}{2}})(1-aq^{-\frac{1}{2}}z^{-1})-(1-az^{-1}q^{n-\frac{1}{2}})(1-aq^{-\frac{1}{2}}z)]}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})\left(z-\frac{1}{z}\right)\frac{1}{2}} \\
&= \frac{\phi_{n-1}(x;aq^{\frac{1}{2}})\left[-aq^{n-\frac{1}{2}}\left(z-\frac{1}{z}\right)+aq^{-\frac{1}{2}}(z-\frac{1}{z})\right]}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})\left(z-\frac{1}{z}\right)\frac{1}{2}} \\
&= \frac{2aq^{-\frac{1}{2}}\phi_{n-1}(x;aq^{\frac{1}{2}})(-q^n+1)}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})} = -2\frac{1-q^n}{1-q}a\phi_{n-1}(x;aq^{\frac{1}{2}}).
\end{aligned}$$

Therefore, we see that $\mathcal{D}_q$ does what we expect it to do to $\phi_n(x;a)$, namely giving us some q-multiple of the Askey–Wilson basis of degree $n-1$. Furthermore, we have

$$\lim_{q\to 1^-}\phi_n(x;a) = \prod_{k=0}^{n-1}(1-2xa+a^2) = (1-2ax+a^2)^n = (-2a)^n\left(x-\frac{a^2+1}{2a}\right)^n,$$

a variant of the classical Taylor basis polynomial centred at $(a^2+1)/(2a)$. We finish this section by formally stating what $\mathcal{D}_q$ does to the Askey–Wilson basis.

Theorem 2.1. *The action of the Askey–Wilson divided difference operator on the Askey–Wilson basis is given by*

$$\mathcal{D}_q\phi_n(x;a) = -\frac{2a(1-q^n)}{1-q}\phi_{n-1}(x;aq^{\frac{1}{2}}).$$

Therefore, more generally, the k-th Askey–Wilson derivative of the Askey–Wilson basis is given by

$$\begin{aligned}
\mathcal{D}_q^k\phi_n(x;a) &= \left(-\frac{2a(1-q^n)}{1-q}\right)\left(-\frac{2aq^{\frac{1}{2}}(1-q^{n-1})}{1-q}\right)\cdots\left(-\frac{2aq^{\frac{k-1}{2}}(1-q^{n-k+1})}{1-q}\right)\phi_{n-k}(x;aq^{\frac{k}{2}}) \\
&= \frac{(2a)^kq^{(0+1+\cdots+(k-1))/2}}{(q-1)^k}\cdot\frac{(q;q)_n}{(q;q)_{n-k}}\phi_{n-k}(x;aq^{\frac{k}{2}}) = \frac{(2a)^kq^{\frac{k(k-1)}{4}}(q;q)_n}{(q-1)^k(q;q)_{n-k}}\phi_{n-k}(x;aq^{\frac{k}{2}}).
\end{aligned}$$

2.3. q-Taylor series for polynomials

Now that we found a workable basis in the previous section, we are only left with finding the coefficient for each of the $\phi_n(x;a)$. That is, if $f(x)$ is a polynomial, then we want to find the appropriate f_k so that we have

$$f(x) = \sum_{k=0}^{n} f_k\phi_k(x;a).$$

Let's take a brief detour to the classical case. Any Taylor polynomial of degree n of $f(x)$ centred at $x=a$ gives us the constant, namely $f(a)$ when $x=a$, i.e., when the classical Taylor basis elements are all zero, and the coefficient of x^k is determined by the values of

$f^{(k)}(a)$. Thus, it is natural to consider for which x_k we have $\phi_k(x_k; a) = 0$. Furthermore, we would expect $\mathcal{D}_q^k f(x_k)$ to show up in f_k in some way.

Recall also that if $f(x) = \sum f_k(x-a)^k$ is the Taylor series of $f(x)$ centred at $x = a$, we can find f_k by differentiation:

$$\frac{d^j}{dx^j} f(x) = j! f_j + \sum_{k=j+1}^{\infty} f_k [k]_j (x-a)^{k-j},$$

where $[k]_j := (k)(k-1)\cdots(k-j+1)$ is the falling factorial. In the classical case, it happens that plugging in $x = a$ uniformly cancels out everything except for $j! f_j$, so indeed we have

$$\frac{d^j}{dx^j} f(a) = j! f_j.$$

However, in the Askey–Wilson case, we need to choose the x value carefully depending on the degree of the Askey–Wilson basis, as we will see shortly, one of the key differences between the classical case and the q-analogue case.

For a polynomial $f(x)$, let

$$f(x) = \sum_{j=0}^{n} f_k \phi_k(x; a),$$

and compute the k-th Askey–Wilson derivative, i.e.,

$$\begin{aligned}
\mathcal{D}_q^k f(x) &= \sum_{j=0}^{n} \mathcal{D}_q^k f_j \phi_j(x; a) \\
&= \sum_{j=k}^{n} f_j \frac{(2a)^k q^{k(k-1)/4} (q;q)_n}{(q-1)^k (q;q)_{n-k}} \phi_{j-k}(x; aq^{\frac{k}{2}}) \\
&= f_k \frac{(2a)^k q^{k(k-1)/4} (q;q)_n}{(q-1)^k (q;q)_{n-k}} + \sum_{j=k+1}^{n} f_j \frac{(2a)^k q^{k(k-1)/4} (q;q)_n}{(q-1)^k (q;q)_{n-k}} \phi_{j-k}(x; aq^{\frac{k}{2}}).
\end{aligned} \tag{2.1}$$

Similar to the classical case, upon substituting x with some appropriate value, we want only the constant term to remain. To do so, we need to find x_k so that $\phi_{j-k}(x; aq^{\frac{k}{2}}) = 0$. If $j = k+1$, then we obtain the equation

$$1 - 2aq^{\frac{k}{2}} x + a^2 q^k = 0,$$

so we have

$$x = \frac{1}{2}\left(aq^{\frac{k}{2}} + \frac{1}{aq^{\frac{k}{2}}} \right) =: x_k.$$

We claim that this x_k also satisfies $\phi_{j-k}(x_k; aq^{\frac{k}{2}}) = 0$ for all $j > k$. Indeed, we have

$$\begin{aligned}
\phi_{j-k}(x_k; aq^{\frac{k}{2}}) &= \prod_{m=0}^{j-k-1} (1 - 2x_k a q^{\frac{k}{2}+m} + a^2 q^{k+2m}) \\
&= \prod_{m=0}^{j-k-1} (1 - a^2 q^{k+m} - q^m + a^2 q^{k+2m})
\end{aligned}$$

$$= \prod_{m=0}^{j-k-1} (1-q^m)(1-a^2q^{k+m}),$$

which is clearly equal to 0 as $1-q^m=0$ when $m=0$. Thus if $x=x_k$, (2.1) becomes

$$\mathcal{D}_q^k f(x_k) = f_k \frac{(2a)^k q^{k(k-1)/4}(q;q)_n}{(q-1)^k(q;q)_{n-k}},$$

so it follows that

$$f_k = \frac{(q-1)^k q^{-k(k-1)/4}(q;q)_{n-k}}{(2a)^k(q;q)_n}\mathcal{D}_q^k f(x_k) = \frac{(q-1)^k q^{-k(k-1)/4}}{(2a)^k(q;q)_k}\mathcal{D}_q^k f(x_k).$$

Our work gives the following theorem, which we formally state.

Theorem 2.2. *Let $x=\cos\theta$. Then for any polynomial $f(x)$, we have*

$$f(x) = \sum f_k\phi_k(x;a),$$

where

$$f_k := \frac{(q-1)^k q^{-k(k-1)/4}}{(2a)^k(q;q)_k}\mathcal{D}_q^k f(x_k)$$

and

$$x_k := \frac{1}{2}\left(aq^{\frac{k}{2}} + \frac{1}{aq^{\frac{k}{2}}}\right).$$

We finish by noting that the Askey–Wilson basis as one possible Taylor basis for the q-analogue. Now we introduce two new basis, both discovered by Ismail and Stanton (as always, $x=\cos\theta$ and $z=e^{i\theta}$).

$$\begin{aligned}
\phi_n(x) &= (q^{1/4}z, q^{1/4}z; q^{1/2})_n \\
\rho_n(x) &= (1+e^{2i\theta})e^{-in\theta}(-q^{2-n}e^{2i\theta};q^2)_{n-1} \\
\rho_0(x) &= 1.
\end{aligned}$$

For the sake of completeness, we will state the q-Taylor theorems for the two Ismail–Stanton basis without proof.

Theorem 2.3. *If $\{\phi_n(x)\}$ and $\{\rho_n(x)\}$ are the Ismail–Stanton bases as defined above, then the Askey–Wilson difference operator acts on these bases in the following manner.*

$$\begin{aligned}
\mathcal{D}_q\phi_n(x) &= -2q^{1/4}\frac{1-q^n}{1-q}\phi_{n-1}(x), \\
\mathcal{D}_q\rho_n(x) &= 2q^{(1-n)/2}\frac{1-q^n}{1-q}\rho_{n-1}(x).
\end{aligned}$$

Theorem 2.4. *Suppose that f is a polynomial of degree n, and let $\zeta_0 := \frac{1}{2}(q^{1/4}+q^{-1/4})$, and let $f_k(\phi)$ (resp. $f_k(\rho)$) be the corresponding coefficient for $\phi_k(x)$ (resp. $\rho_k(x)$ in the q-Taylor series of f with respect to $\{\phi_n(x)\}$ (resp. $\{\rho_n(x)\}$). Then we have*

$$\begin{aligned}
f_k(\phi) &= \frac{(q-1)^k q^{-k/4}}{2^k(q;q)_k}(\mathcal{D}_q^k f)(\zeta_0) \\
f_k(\rho) &= \frac{(1-q)^k q^{k(k-1)/4}}{2^k(q;q)_k}(\mathcal{D}_q^k f)(0).
\end{aligned}$$

3. q-Hypergeometric functions and q-summation theorems (Analysis)

3.1. q-hypergeometric functions

In the previous section, we established that the Askey–Wilson operator, and that the Askey–Wilson basis behaves in a way that the classical Taylor basis counterpart (i.e., $(x-a),(x-a)^2,\dots)$ with respect to the Askey–Wilson operator. Now we present some examples of the q-Taylor series for polynomials. Before presenting the example, we will introduce a notation that will be used throughout this section and this lecture note.

Definition 3.1. The *q-hypergeometric function* is defined by

$$\begin{aligned} {}_r\phi_s\left(\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \middle| q, z\right) &= {}_r\phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z) \\ &:= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n} z^n \left(-q^{\binom{n}{2}}\right)^{s-r+1}. \end{aligned}$$

The *generalized hypergeometric function* is defined by

$${}_rF_s\left(\begin{matrix} a_1, \ \dots a_r \\ b_1, \ \dots, \ b_s \end{matrix} \middle| z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r)_n}{(1,b_1,\dots,b_s)_n} z^n,$$

where $(a_1,\dots,a_r)_n$ denotes the *multi-shifted factorial*, i.e.,

$$(a_1,\dots,a_r)_n := \prod_{j=1}^{n} (a_j)_n = \prod_{j=1}^{n} a_j(a_j+1)\cdots(a_j+n-1).$$

If $n=0$, then $(a)_0 := 1$.

Remark. If one of the numerator parameters is of the form q^{-k}, then the infinite sum in the definition of the q-hypergeometric function terminates at k.

3.2. q-summation theorems

Let $f(x)=\phi_n(x;b)$, and we want to find the q-Taylor series for $f(x)$ centred at a. Indeed, we have

$$\mathcal{D}_q\phi_n(x;b) = -\frac{2b(1-q^k)}{1-q}\phi_{n-1}(x;bq^{\frac{1}{2}}).$$

So in general, we have, by virtue of Theorem 2.1,

$$\begin{aligned} \mathcal{D}_q^k\phi_n(x;b) &= \frac{(2b)^k(1-q)^n\cdots(1-q^{n-k+1})q^{\frac{k(k-1)}{4}}}{(q-1)^k}\phi_{n-k}(x;bq^{\frac{k}{2}}) \\ \therefore\ \mathcal{D}_q^k\phi_n(x_k;b) &= \frac{(2b)^k(1-q)^n\cdots(1-q)^{n-k+1}q^{\frac{k(k-1)}{4}}}{(q-1)^k}\left(bq^{k/2}aq^{k/2}, \frac{bq^{k/2}}{aq^{k/2}};q\right)_{n-k} \\ &= \frac{(2b)^k(1-q^n)\cdots(1-q^{n-k+1})q^{\frac{k(k-1)}{4}}}{(q-1)^k}\left(abq^k, \frac{b}{a};q\right)_{n-k}, \end{aligned}$$

so by Theorem 2.2, it we have

$$f_k = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}\left(\frac{b}{a}\right)^k\left(abq^k, \frac{b}{a};q\right)_{n-k}.$$

On the other hand,

$$\begin{aligned}
(bz, b/z; q)_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{b}{a}\right)^k \frac{\left(\frac{b}{a}; q\right)_{n-k} (ab;q)_n}{(ab;q)_k} \left(az, \frac{a}{z}; q\right)_k \\
&= (ab;q)_n (q;q)_n \sum_{k=0}^{n} \frac{(b/a)^k}{(ab, q; q)_k} \cdot \frac{\left(\frac{b}{a}; q\right)_{n-k}}{(q;q)_{n-k}} \left(az, \frac{a}{z}; q\right)_k \\
&\overset{*}{=} \sum_{k=0}^{n} \frac{(ab, q; q)_n (\frac{b}{a}; q)_n}{(q;q)_n (q, ab; q)_k} \cdot \frac{(q^{-n}; q)_k}{\left(\frac{q^{1-n}a}{b}; q\right)_k} \left(\frac{qa}{b}\right)^k \left(\frac{b}{a}\right)^k \left(az, \frac{a}{z}; q\right)_k \\
&= {}_3\phi_2\left(\begin{matrix} q^{-n}, az, \frac{a}{z} \\ ab, \frac{q^{1-n}a}{b} \end{matrix} \,\middle|\, q, q\right) (ab;q)_n \left(\frac{b}{a}; q\right)_n .
\end{aligned} \tag{$\dagger$}$$

with the $\overset{*}{=}$ following from the identity

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}} = \frac{(a;q)_n \left(\frac{q^{1-n}}{b}; q\right)_k}{(b;q)_n \left(\frac{q^{1-n}}{a}; q\right)_k} \left(\frac{b}{a}\right)^k .$$

Our work above gives rise to the following theorem.

Theorem 3.1 (q-Pfaff-Saalschütz theorem)**.** *For $cd = abq^{1-n}$,*

$${}_3\phi_2\left(\begin{matrix} q^{-n}, a, b \\ c, d \end{matrix} \,\middle|\, q, q\right) = \frac{(d/a, d/b; q)_n}{(d, d/(ab); q)_n}.$$

Definition 3.2. We say $f(z)$ is *analytic at z_0* if f is differentiable for all $z \in B_r(z_0)$ for some $r > 0$.

Proposition 3.1. *If f is analytic at z_0, then all of its derivatives are also analytic at z_0. Furthermore, f can be expressed in the Taylor series for all $z \in B_r(z_0)$, i.e., there exist coefficients f_k so that*

$$f(x) = \sum_{k=0}^{\infty} f_k (z - z_0)^k$$

for all $z \in B_r(z_0)$.

Definition 3.3. The points where f is not analytic are called *singularities.*

Example. Consider the following function

$$f(z) = {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q, z\right) = \sum_{n=0}^{\infty} a_n z^n$$

where

$$a_n := \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n}.$$

We see that as long as $|z| < 1$, the series converges, making it analytic, since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(a;q)_{n+1}(b;q)_{n+1}}{(c;q)_{n+1}(q;q)_{n+1}} \frac{(c;q)_n (q;q)_n}{(a;q)_n (b;q)_n}$$

$$= \lim_{n\to\infty} \frac{(1-aq^n)(1-bq^n)}{(1-cq^n)(1-q^{n+1})} = 1.$$

So for any $|z| < 1$,

$$\begin{aligned}
(1-z)f(z) &= \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n z^{n+1} \\
&= a_0 + \sum_{n=0}^{\infty} a_{n+1} z^{n+1} - \sum_{n=0}^{\infty} a_n z^{n+1} \\
&= a_0 + \sum_{n=0}^{\infty} z^{n+1}(a_{n+1} - a_n) \\
&= a_0 + \sum_{n=0}^{\infty} z^{n+1} \frac{(a;q)_n (b;q)_n}{(c;q)_{n+1}(q;q)_{n+1}} \left[(1-aq)^n(1-bq^n) - (1-cq^n)(1-q^{n+1})\right].
\end{aligned}$$

But we can further simplify upon recognizing that

$$(1-aq^n)(1-bq^n) - (1-cq^n)(1-q^{n+1}) = q^n(-a-b+c+q) + q^{2n}(ab-cq),$$

which yields

$$(1-z)f(z) = a_0 + z\sum_{n=0}^{\infty} z^n q^n C_n,$$

where

$$C_n := \frac{(a;q)_n (b;q)_n}{(c;q)_{n+1}(q;q)_{n+1}}[(-a-b+c+q) + q^n(ab-cq)]$$

which we know is bounded. This shows that $(1-z)f(z)$ is analytic for $|qz| < 1$.

The example above proves the following theorem.

Theorem 3.2. $(c;q)_\infty (z;q)_\infty \, {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| \, q, z\right)$ *is analytic for all c and all $z \in \mathbb{C}$ except for the point at infinity.*

Theorem 3.3 (q-Chu-Vandermonde sum)**.**

$${}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix} \middle| \, q, q\right) = \frac{(c/a;q)_n}{(c;q)_n} a^n.$$

Proof. Start with the expression from Theorem 3.1. Letting $b \to 0$ gives

$${}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix} \middle| \, q, q\right) = \lim_{b\to 0} \frac{(c/a;q)_n}{(c;q)_n} \frac{(b-c)(b-qc)\cdots}{\left(b-\frac{c}{a}\right)\left(b-\frac{c}{a}q\right)\cdots} = \frac{(c/a;q)_n}{(c;q)_n} a^n. \qquad \square$$

Recall that we can exchange the limit operator and the summand, i.e.,

$$\lim_{n\to\infty} \sum_{k=0}^{n} f_k(n) = \sum_{k=0}^{\infty} \lim_{n\to\infty} f_k(n)$$

only under certain circumstances, and the following theorem illustrates when we are permitted to do so.

Theorem 3.4 (Tannery's theorem)**.** *Suppose* $\{f_k(z,n)\}_n$ *is a sequence such that* $f_k(z,n) \to f_k(z)$ *as* $n \to \infty$*. If* $|f_k(z,n)| \le M_k$ *for some* M_k*, and* $\sum M_k < \infty$*, then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} f_k(z,n) = \sum_{k=0}^{\infty} f_k(z).$$

With this in mind, we will prove the q-analogue of the Gauss summation theorem (also known as the Gauss hypergeometric theorem).

Theorem 3.5 (q-Gauss summation theorem)**.** *Provided* $|c/(ab)| < 1$*, we have*

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle|\, q, \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/(ab); q)_\infty}.$$

Proof. To prove this, we will compute the limit as $n \to \infty$ in ${}_3\phi_2$ from Theorem 3.1. Note that such operation (taking the limit inside the summation) is justified by Tannery's theorem since $(a,b;q)_k/(q,c;q)_k$ is bounded for all $|q| < 1$.

$$\begin{aligned}
\frac{(q^{-n};q)_k}{(q^{1-n}(ab)/c;q)_k} &= \frac{(1-q^{-n})\cdots(1-q^{-n+k-1})}{(1-\frac{ab}{c}q^{1-n})\cdots(1-\frac{ab}{c}q^{1-n+k-1})} \\
&= \frac{q^{k(k-1)/2}c^k}{(ab)^k q^{k(k+1)/2}} \cdot \frac{(1-q^n)\cdots(1-q^{n-k})}{(1-\frac{c}{ab}q^{n-1})\cdots(1-\frac{c}{ab}q^{n-k})} \\
&= \left(\frac{c}{ab}\right)^k \frac{1}{q^k} \frac{(q;q)_n}{(q;q)_{n-k}} \frac{(\frac{c}{ab};q)_{n-k}}{(\frac{c}{ab};q)_n} \to \left(\frac{c}{abq}\right)^k \cdot 1,
\end{aligned}$$

as $n \to \infty$. Hence, as $n \to \infty$ we get

$$\sum_{k=0}^{\infty} \frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k}\left(\frac{c}{ab}\right) = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/(ab);q)_\infty}.$$ □

From this, the classical Gauss summation theorem readily follows.

Corollary 3.1 (Gauss summation theorem)**.** *For all* $\mathrm{Re}(C-A-B) > 0$*,*

$${}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \middle|\, 1\right) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)},$$

Proof. Let a, b, and c be as they appear from Theorem 3.5. let $a = q^A, b = q^B, c = q^C$. Then Theorem 3.5 becomes

$$\begin{aligned}
{}_2\phi_1\left(\begin{matrix} q^A, q^B \\ q^C \end{matrix} \middle|\, q, q^{C-A-B}\right) &= \frac{(q^{C-A}, q^{C-B};q)_\infty}{(q^C, q^{C-A-B};q)_\infty} \\
&= \left(\frac{(q;q)_\infty}{(q^C;q)_\infty}\right)\left(\frac{(q;q)_\infty}{(q^{C-A-B};q)_\infty}\right)\left(\frac{(q^{C-A};q)_\infty}{(q;q)_\infty}\right)\left(\frac{(q^{C-B};q)_\infty}{(q;q)_\infty}\right) \\
&\qquad \times \frac{(1-q)^{1-C}(1-q)^{1-C+A+B}}{(1-q)^{1-C+A}(1-q)^{1-C+B}} \\
&= \frac{\Gamma_q(C)\Gamma_q(C-A-B)}{\Gamma_q(C-A)\Gamma_q(C-B)}.
\end{aligned}$$

Also, note that $q \to 1^-$ gives:

$$\frac{(a;q)_k}{(1-q)^k} = \frac{1-q^A}{1-q}\frac{1-q^{A-1}}{1-q}\cdots\frac{1-q^{A+k-1}}{1-q} \to A(A+1)\cdots(A+k-1).$$

Hence the limit is

$$\sum_{k=0}^{\infty}\frac{A(A+1)\cdots(A+k-1)B(B+1)\cdots(B+k-1)}{C(C+1)\cdots(C+k-1)k!}1^k = {}_2F_1\left(\begin{matrix}A,B\\C\end{matrix}\,\middle|\,1\right).$$

Furthermore, $\Gamma_q(z) \to \Gamma(z)$ as $q \to 1^-$, so indeed it follows that

$${}_2F_1\left(\begin{matrix}A,B\\C\end{matrix}\,\middle|\,1\right) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}.$$

□

Corollary 3.2. *If $|z|<1$ or $a=q^{-n}$, then*

$${}_1\phi_0\left(\begin{matrix}a\\ \cdot\end{matrix}\,\middle|\,q,x\right) = \frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k,$$

where $|x|<1$.

Proof. Write $c=abx$ in Theorem 3.5, and then let $b\to 0$. The second equality follows from the general q-binomial theorem. □

We finish with the following summation theorem proved by Ismail and Stanton. Since the proof involves the Ismail–Stanton ϕ-basis, the proof is beyond the scope of this lecture note.

Theorem 3.6. $\dfrac{(az,a/z;q^2)_n}{(aq^{-1/2};q)_{2n}} = {}_4\phi_3\left(\begin{matrix}q^{-n},-q^{-n},q^{1/2}z,q^{1/2}z\\ -q,aq^{-1/2},\frac{-q^{2n+\frac{3}{2}}}{a}\end{matrix}\,\middle|\,q,q\right).$

3.3. **Sears transformation and special cases due to Euler**

Theorem 3.7 (Sears transformation)**.**

$$\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,\frac{a}{z}\\ ab,ac,ad\end{matrix}\,\middle|\,q,q\right) = \frac{(ba,bc,bd;q)_n}{b^n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,bz,\frac{b}{z}\\ ba,bc,bd\end{matrix}\,\middle|\,q,q\right)$$

Proof. From (†) we have

$$\frac{(bz,b/z;q)_m}{(q;q)_m} = \sum_{k=0}^{m-1}\frac{b^k}{a^k}\frac{(az,a/z;q)_k(abq^k,b/a;q)_{m-k}}{(q;q)_k(q;q)_{m-k}}$$

But since $(abq^k;q)_{m-k} = (ab;q)_m/(ab;q)_k$, we have

$${}_4\phi_3\left(\begin{matrix}q^{-n},abcdq^{n-1},bz,b/z\\ ab,ac,ad\end{matrix}\,\middle|\,q,q\right) = \sum_{n\geq m\geq k\geq 0}\frac{(q^{-n},abcdq^{n-1};q)_m q^m}{(q,ab;q)_k(ac,ad;q)_m}\frac{b^k}{a^k}.$$

Now change the variable from m to $m+k$. Now the RHS of the above can be re-written as follows.

$$\sum_m \frac{b^k}{a^k}\cdot\frac{(q^{-n},abcdq^{n-1};q)_k}{(q,ab,bc,ad;q)_k}q^k\sum_k\frac{(q^{-n+k},abcdq^{n-1};q)_m(b/a;q)_m}{(q^kbc,q^kbd,q;q)_m}q^m,$$

which is equal to

$$ {}_3\phi_2\left(\begin{matrix} q^{-n+k}, abcdq^{n-k-1}, b/a \\ q^k bc, q^k bd \end{matrix} \,\middle|\, q, q\right). $$

The claim follows upon noting that, indeed

$$ \frac{b}{a} q^{-n+k+1} abcdq^{n+k-1} = q^k bcq^k bd. \qquad \square $$

Now we present a few special cases from Euler.

Theorem 3.8 (Euler). $e_q(x) := \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}.$

Proof. Start with Theorem 3.7, and apply $a = 0$. $\square$

Theorem 3.9 (Euler). $E_q(z) := (-z;q)_\infty = \displaystyle\lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \frac{z^n}{a^n} = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} q^{\binom{n}{2}}$

Proof. Let $x = -z/a$, and let $a \to \infty$ as it appears from Theorem 3.7.

$$ \lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(q;q)_n} \frac{(-z)^n}{qa^n} = \sum_n \frac{z^n}{(q;q)_n} q^{\binom{n}{2}}. \qquad \square $$

Corollary 3.3. $\displaystyle\lim_{q\to 1^-} e_q((1-q)x) = e^x$ *and* $\displaystyle\lim_{q\to 1^-} E_q((1-q)x) = e^x$.

4. Ramanujan ${}_1\psi_1$ sum

The main goal in this section is to evaluate the Ramanujan ${}_1\psi_1$ sum. Before stating Ramanujan's theorem, we first define ${}_m\psi_m$.

Definition 4.1. A *bilateral hypergeometric function* is defined by

$$ {}_m\psi_m\left(\begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_m \end{matrix} \,\middle|\, q, z\right) := \sum_{n=-\infty}^{\infty} \frac{(a_1, \ldots, a_m; q)_n}{(b_1, \ldots, b_m; q)_n} z^n. $$

We also recall one more theorem from complex analysis that we will need in the final step of the theorem. Recall that for the notion of analyticity and holomorphicity coincide in complex analysis.

Theorem 4.1 (Identity theorem for holomorphic functions). *Assume that f and g are analytic on a domain D, and $\{z_n\}$ is a sequence that converges to $a \in D$. If $f(z_n) = g(z_n)$, then $f \equiv g$ for all points in D.*

Now we state our main theorem.

Theorem 4.2 (Ramanujan ${}_1\psi_1$ sum). *Suppose $|b/a| < |z| < 1$. Then we have*

$$ {}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix} \,\middle|\, q, z\right) = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(b/a, q, q/(az), az; q)_\infty}{(b, b/(az), q/a, z; q)_\infty}. $$

The following proof was given by Ismail in [Ism77].

Proof. Recall that, for any $n > 0$,

$$(c;q)_{-n} = \frac{1}{(cq^{-n};q)_n}. \tag{4.1}$$

This observation will come handy in proving Theorem 4.2. As a function of b, $(b;q)_n$ is analytic in b provided that $|b| < 1$ and $n \geq 0$. $(b;q)_n$ is also analytic when $n < 0$. Also, note that ${}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix} \,\middle|\, q, z\right)$ is analytic as a function of b for $|b| < |az|$. Indeed, by our observation (4.1), we can rewrite the given Ramanujan ${}_1\psi_1$ sum as follows.

$$\begin{aligned}
{}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix} \,\middle|\, q, z\right) &= \sum_{n=-\infty}^{-1} \frac{(a;q)_n}{(b;q)_n} z^n + \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n \\
&= \sum_{n=1}^{\infty} \frac{(bq^{-n};q)_n}{(aq^{-n};q)_n} \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n \\
&= \sum_{n=1}^{\infty} \frac{(1-bq^{-n})\cdots(1-bq^{-1})}{(1-aq^{-n})\cdots(1-aq^{-1})} \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n \\
&= \sum_{n=1}^{\infty} \frac{(b^{-1}-q^{-n})\cdots(b^{-1}-q^{-1})}{(a^{-1}-q^{-n})\cdots(a^{-1}-q^{-1})} \left(\frac{b}{az}\right)^n + \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n.
\end{aligned}$$

But since

$$b^{-1} - q^{-k} = -q^{-k}(1 - q^k b^{-1}),$$

we have

$$\frac{(b^{-1}-q^{-n})\cdots(b^{-1}-q^{-1})}{(a^{-1}-q^{-n})\cdots(a^{-1}-q^{-1})} = \frac{(-1)^n q^{-\binom{n+1}{2}}(1-b^{-1}q^n)\cdots(1-b^{-1}q)}{(-1)^n q^{-\binom{n+1}{2}}(1-a^{-1}q^n)\cdots(1-a^{-1}q)} = \frac{(qb^{-1};q)_n}{(qa^{-1};q)_n}.$$

Thus all in all, we have

$${}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix} \,\middle|\, q, z\right) = \sum_{n=1}^{\infty} \frac{(qb^{-1};q)_n}{(qa^{-1};q)_n} \left(\frac{b}{az}\right)^n + \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n,$$

so indeed we need $|b| < |az|$ and $|b| < 1$ in order for the sum to be analytic.

If we let $b = q^{m+1}$ for $m = 0, 1, 2, \ldots$, then for $n > 0$ we have

$$\frac{1}{(b;q)_{-n}} = (bq^{-n};q)_n = (q^{m+1-n};q)_n = (1-q^{m+1-n})\cdots(1-q^m),$$

so if $n > m$ then the product is always 0. So it follows that

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(q^{m+1};q)_n} z^n = \sum_{n=-m}^{\infty} \frac{(a;q)_n}{(q^{m+1};q)_n} z^n = \sum_{k=0}^{\infty} \frac{(a;q)_{-m+k}}{(q^{m+1};q)_{-m+k}} z^{-m+k}.$$

with the last equality following by letting $n = -m + k$. Since $(\lambda;q)_{j+s} = (\lambda;q)_j(\lambda q^j;q)_s$ for all j and s, we now have

$$\sum_{k=0}^{\infty} \frac{(a;q)_{-m+k}}{(q^{m+1};q)_{-m+k}} z^{-m+k} = \frac{(a;q)_{-m}}{(q^{m+1};q)_{-m}} z^{-m} \sum_{k=0}^{\infty} \frac{(aq^{-m};q)_k}{(q;q)_k} z^k$$

$$
\begin{aligned}
&= z^{-m}\frac{(a;q)_{-m}}{(q^{m+1};q)_{-m}}\cdot\frac{(aq^{-m}z;q)_\infty}{(z;q)_\infty} \text{ (by the q-binomial theorem)}\\
&= \frac{z^{-m}(a;q)_{-m}}{(q^{m+1};q)_{-m}}\frac{(aq^{-m}z;q)_m}{(z;q)_\infty}(az;q)_\infty\\
&= z^{-m}\frac{(q;q)_m(azq^{-m};q)_m}{(q^{-m}a;q)_m(z;q)_\infty}(az;q)_\infty.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{(azq^{-m};q)_m}{(aq^{-m};q)_m} &= \frac{(1-azq^{-m})\cdots(1-azq^{-1})}{(1-aq^{-m})\cdots(1-aq^{-1})}\\
&= z^m\frac{(1-\frac{q}{az})(1-\frac{q^2}{az})\cdots(1-\frac{q^m}{az})}{(1-\frac{q}{a})\cdots(1-\frac{q^m}{az})}\\
&= z^m\frac{(\frac{q}{az};q)_\infty(\frac{q^{m+1}}{a};q)_\infty}{(\frac{q^{m+1}}{az};q)_\infty(\frac{q}{a};q)_\infty},
\end{aligned}
$$

so

$$
z^{-m}\frac{(q;q)_m(azq^{-m};q)_m}{(q^{-m}a;q)_m(z;q)_\infty}(az;q)_\infty = \frac{(\frac{q}{az};q)_\infty(\frac{b}{a};q)_\infty}{(\frac{q}{a},\frac{b}{az};q)_\infty}\frac{(q;q)_m(az;q)_\infty}{(z;q)_\infty} = \frac{(\frac{q}{az},\frac{b}{a},az,q;q)_\infty}{(\frac{q}{a},\frac{b}{az},b,z;q)_\infty}.
$$

This proves that the Ramanujan ${}_1\psi_1$ sum is equal to the right side of the equation for all $b = q^{m+1}$. But since $|q| < 1$, the sequence $\{q^{m+1}\}$ converges to 0, which is clearly inside the unit disc. Therefore the equality holds for every $|b| < 1$ and $|b| < |az|$ by Theorem 4.1. $\square$

We also note that the Ramanujan ${}_1\psi_1$ sum includes the Jacobi triple product.

Theorem 4.3 (Jacobi triple product)**.** $\displaystyle\sum_{n=-\infty}^{\infty} q^{n^2}z^n = (q^2;q^2)_\infty(-qz^{-1};q^2)_\infty(-qz;q^2)_\infty.$

Proof. Apply Theorem 4.2 to the following identity first. Note $(0;q)_n = 1$ for any $n \in \mathbb{Z}\cup\{\infty\}$, so

$$
{}_1\psi_1\left(\begin{matrix}-1/c\\ 0\end{matrix}\,\middle|\, q^2, qzc\right) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{c};q^2\right)_n (qzc)^n = \frac{(q^2,-qz^{-1},-qz;q^2)_\infty}{(-q^2c,qzc;q)_\infty}. \tag{4.2}
$$

By Tannery's theorem, we may apply the limit $c \to 0$ inside the infinite sum.

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty}\lim_{c\to 0}\left(-\frac{1}{c};q^2\right)_n(qzc)^n &= \sum_{n=-\infty}^{\infty}\lim_{c\to 0}(1+c^{-1})(1+c^{-1}q^2)\cdots(1+c^{-1}q^{2(n-1)})(qzc)^n\\
&= \sum_{n=-\infty}^{\infty}\lim_{c\to 0}c^{-n}(c+1)(c+q^2)\cdots(c+q^{2n-2})q^nc^nz^n\\
&= \sum_{n=-\infty}^{\infty}\lim_{c\to 0}(c+q)(c+q^3)\cdots(c+q^{2n-1})z^n\\
&= \sum_{n=-\infty}^{\infty} q^{1+3+\cdots+(2n-1)}z^n = \sum_{n=-\infty}^{\infty}q^{n^2}z^n.
\end{aligned} \tag{4.3}
$$

On the other hand, we have

$$\lim_{c\to 0}\frac{(q^2,-qz^{-1},-qz;q^2)_\infty}{(-q^2c,qzc;q)_\infty}=(q^2;q^2)_\infty(-qz^{-1};q^2)_\infty(-qz;q^2)_\infty. \tag{4.4}$$

Putting (4.3) and (4.4) together yields us

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n=(q^2;q^2)_\infty(-qz^{-1};q^2)_\infty(-qz,q^2)_\infty. \qquad \square$$

5. More on the Askey–Wilson calculus

We will further derive the Askey–Wilson counterpart of some calculus rules (e.g. product rule, Leibniz rule). For the sake of completeness, we will state that the product rule and the Leibniz rule from the classical calculus.

Theorem 5.1. *If $f,g:\mathbb{R}\to\mathbb{R}$ are C^n (i.e., n times differentiable functions), we have*

$$\frac{d}{dx}(f(x)g(x))=f'(x)g(x)+f(x)g'(x)$$
$$\frac{d^n}{dx^n}(f(x))g(x))=\sum_{j=0}^{n}\binom{n}{k}f^{(n-k)}(x)g^{(k)}(x).$$

Definition 5.1. Let $\mathcal{A}_q$ be the *Askey–Wilson average operator* defined by

$$\mathcal{A}_qf(x)=\frac{\breve{f}(q^{1/2}z)+\breve{f}(q^{-1/2}z)}{2}.$$

Theorem 5.2 (Askey–Wilson product rule)**.** *$\mathcal{D}_q(fg)=\mathcal{A}_qf\mathcal{D}_qg+\mathcal{D}_qf\mathcal{A}_qg$.*

More generally, Ismail proved the Leibniz rule counterpart for $\mathcal{D}_q$:

Theorem 5.3 (Askey–Wilson Leibniz rule)**.** *We have*

$$\mathcal{D}_q^n(fg)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-n)/2}\mathcal{D}_q^{n-k}f(q^{k/2}z)\mathcal{D}_q^kg(q^{-(n-k)/2}z).$$

Cooper proved the nth Askey–Wilson derivative formula.

Theorem 5.4 (Cooper)**.** *The nth Askey–Wilson derivative of f is*

$$\mathcal{D}_q^nf(x)=\frac{2^nq^{n(1-n)/4}}{(q^{1/2}-q^{-1/2})^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{k(n-k)}z^{2k-n}\breve{f}(q^{(n-2k)/2}z)}{(q^{1+n-2k}z^2;q)_k(q^{2k-n+1}z^{-2};q)_{n-k}}.$$

6. q-Gamma function

In this section we explore the q-analogue of the gamma function.

6.1. **Classical gamma function**

Definition 6.1. The *(classical) gamma function* is defined by

$$\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}\,dx$$

for all $\mathrm{Re}(z) > 0$.

The gamma function is one of the functions such that $f(1) = 1$ and $f(x+1) = xf(x)$ for all positive real number x, leading us to conclude that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. However, with just two aforementioned restrictions, one can generate infinitely many functions of such kind by multiplying any periodic analytic functions that hits 1 for any positive integer. However, with one additional restriction called log-convexity, the solution is unique: it turns out that the gamma function is the only function satisfying all three properties. This theorem is known as the Bohr-Mollerup theorem. This prompt us to introduce the following definition.

Definition 6.2. A function f is said to be *convex* on (a, b) if f is positive in (a, b), and for all $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

If $\log f$ is convex, then f is said to be *log-convex* or *logarithmically convex*.

We state the Bohr–Mollerup theorem for the sake of comparing with the q-analogue case which we will explore in the remainder of this section.

Theorem 6.1 (Bohr–Mollerup theorem)**.** *The gamma function* $\Gamma(z)$ *is the* unique function f *on* $\mathrm{Re}(z) > 1$ *satisfying all three properties listed below.*

(1) $f(1) = 1$*,*
(2) $f(x+1) = xf(x)$ *for all positive* $x \in \mathbb{R}$*, and*
(3) f *is log-convex.*

6.2. q-**gamma function**

We want to find Γ_q satisfying all the properties that the regular Γ has, namely:

(1) $\Gamma_q(1) = 1$
(2) $\Gamma_q(z+1) = \dfrac{1-q^z}{1-q}\Gamma_q(z)$.

Note that the second condition implies that, for any $n \in \mathbb{N}$,

$$\Gamma_q(n+1) = \frac{1-q^{n+1}}{1-q}\Gamma_q(n) = \frac{(1-q^{n+1})(1-q^n)}{(1-q)^2}\Gamma_q(n-1) = \cdots = \frac{(q;q)_n}{(1-q)^n}.$$

It is not obvious that such Γ_q is unique. (In fact, with just these two conditions, the answer is no, just like in the classical case.) But again as with the classical case, with the log convexity restriction added, it turns out that Γ_q is the unique function satisfying all three conditions. We first start with a simple lemma regarding convexity.

Lemma 6.1. *Suppose that* f *is convex on* (a, b)*, and suppose that* $0 < x < y < z < b$*. Then*

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}.$$

Proof. Let $y = \lambda x + (1-\lambda)z$ with $\lambda \in (0,1)$. Then we have

$$\frac{f(y)-f(x)}{y-x} \leq \frac{\lambda f(x)+(1-\lambda)f(z)-f(x)}{\lambda x+(1-\lambda)z-x} = \frac{(1-\lambda)(f(z)-f(x))}{(1-\lambda)(z-x)} = \frac{f(z)-f(x)}{z-x},$$

so this proves the first inequality. One can employ the similar type of argument for the second inequality. □

Askey proved in 1978 the q-gamma counterpart of the Bohr-Mollerup theorem, which provided the complete characterization of the gamma function, and that such function must be unique.

Theorem 6.2 ("q-Bohr–Mollerup theorem", proved by Askey (1978))**.** *If* $\log f$ *is a convex function such that*

$$f(z+1) = \frac{1-q^z}{1-q} f(z) \text{ and } f(1) = 1,$$

then

$$f(z) = \frac{(1-q)^{1-z}(q;q)_\infty}{(q^z;q)_\infty} =: \Gamma_q(z).$$

Proof. Let $0 < x < 1$ so that we have $n < n+1 < n+1+x < n+2$. By Lemma 6.1, we have

$$\log f(n+1) - \log f(n) \leq \frac{\log f(n+1+x) - \log f(n+1)}{x} \leq \log f(n+2) - \log f(n+1).$$

Hence

$$\log\left(\frac{1-q^n}{1-q}\right) \leq \frac{1}{x}\left(\log \frac{f(n+1+x)(1-q)^n}{(q;q)_n}\right) \leq \frac{1-q^{n+1}}{1-q}. \tag{6.1}$$

But then we have

$$f(n+1+x) = \frac{1-q^{n+x}}{1-q} \cdots \frac{1-q^x}{1-q} f(x) = \frac{(q^x;q)_{n+1}}{(1-q)^{n+1}},$$

so (6.1) becomes

$$\log \frac{1-q^n}{1-q} \leq \frac{1}{x} \log \frac{(q^x;q)_{n+1} f(x)}{(q;q)_n (1-q)} \leq \log \frac{1-q^{n+1}}{1-q}.$$

Now letting $n \to \infty$ gives us

$$-\log(1-q) \leq \frac{1}{x} \log \frac{(q^x;q)_\infty f(x)}{(q;q)_\infty (1-q)} \leq -\log(1-q).$$

In conclusion, we have

$$f(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{-x+1} =: \Gamma_q(x).$$

□

Theorem 6.3 (Legendre duplication formula)**.** *The q-gamma function satisfies the Legendre duplication formula, i.e.,*

$$\Gamma_q(z) = \frac{\Gamma_{q^2}(\frac{z}{2})\Gamma_{q^2}(\frac{z}{2}+\frac{1}{2})}{\Gamma_{q^2}(\frac{1}{2})(1+q)^{1-z}}.$$

Proof. Notice that both sides are log-convex since all of $\Gamma_q(z), \Gamma_{q^2}(z/2), \Gamma_{q^2}((z+1)/2), (1+q)^{1-z}$ are log-convex as well. Also, if $z = 1$, then both sides are 1 as well. Let the right-hand side be $f(z)$. Then we have

$$\begin{aligned}\frac{f(z+1)}{f(z)} &= \frac{\Gamma_{q^2}(\frac{z+1}{2})\Gamma_{q^2}(\frac{z+2}{2})(1+q)}{\Gamma_{q^2}(\frac{z}{2})\Gamma_{q^2}(\frac{z+1}{2})} \\ &= \frac{1-(q^2)^{z/2}}{1-q^2}(1+q) = \frac{1-q^z}{1-q},\end{aligned}$$

so the claim follows from the uniqueness of the q-gamma function. $\square$

Theorem 6.4 (Generalized Legendre duplication formula)**.** *For any $n \in \mathbb{N}$, we have the following duplication formula.*

$$\Gamma_q(z) = \frac{\Gamma_{q^n}(\frac{z}{n})\Gamma_{q^n}(\frac{z}{n}+\frac{1}{n})\cdots\Gamma_{q^n}(\frac{z}{n}+\frac{n-1}{n})}{\Gamma_{q^n}(\frac{1}{n})\Gamma_{q^n}(\frac{2}{n})\cdots\Gamma_{q^n}(\frac{n-1}{n})}\left(\frac{1-q^n}{1-q}\right)^{z-1}.$$

Proof. We imitate the proof of the Bohr-Mollerup theorem. Suppose that $f(z)$ is equal to the RHS. We need to verify that f satisfies the functional equation satisfied by $\Gamma_q(z)$. Again, $\log f$ is convex since each of $\Gamma_{q^n}(\frac{z}{n}+\frac{i}{n})$ is for $0 \le i \le n-1$. Clearly, $f(1) = 1$. Furthermore,

$$\begin{aligned}f(z+1) &= \frac{\Gamma_{q^n}(\frac{z}{n}+1)}{\Gamma_{q^n}(\frac{z}{n})}f(z)\frac{1-q^n}{1-q} \\ &= \frac{(1-(q^n)^{\frac{z}{n}})}{1-q^n}f(z)\frac{1-q^n}{1-q} = \frac{1-q^z}{1-q}f(z).\end{aligned}$$

So f satisfies all the functional equations, so $f \equiv \Gamma_q$ as required. $\square$

6.3. q-integrals

For this section, we will return to the Jackson operator. Recall that

$$D_q f(x) := \frac{f(x)-f(qx)}{x-qx},$$

so it is straightforward to see that

$$D_q x^n = \frac{1-q^n}{1-q}x^{n-1}.$$

Therefore, as $q \to 1^-$ we see that

$$\lim_{q\to 1^-} D_q = \frac{d}{dx},$$

the classical differential operator from the first-year calculus courses.

Definition 6.3. The *q-integral* is defined by

$$\begin{aligned}\int_0^a f(x)\,d_qx &:= f(a)(a-aq)+f(aq)(aq-aq^2)+\cdots+f(aq^n)(aq^n-aq^{n+1})+\cdots \\ &= a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n).\end{aligned}$$

$$\int_a^b f(x)\,d_qx := \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx.$$

As with the differential operator, we would expect that letting $q \to 1^-$ yields us the classic integral. In fact, Fermat's work showed such is the case for x^m. Indeed,

$$\int_0^a x^m \, d_q x = a(1-q)\sum_{n=0}^{\infty} q^n (aq^n)^m = a^{m+1}(1-q)\sum_{n=0}^{\infty} q^{n(m+1)} = \frac{a^{m+1}(1-q)}{1-q^{m+1}},$$

so $q \to 1^-$ gives us $\frac{a^{m+1}}{m+1}$, which is what we would expect.

Theorem 6.5 (q-integration by parts)**.** *For any a,*

$$\int_0^a (D_q f(x)) g(x) \, d_q x = f(a)g(aq^{-1}) - f(0)g(0) - \frac{1}{q}\int_0^a f(x) D_{q^{-1}} g(x) \, d_q x.$$

Proof. We start from the definition.

$$\int_0^a (D_q f(x)) g(x) \, d_q x = a(1-q)\lim_{m\to\infty}\sum_{n=0}^{m} q^n \frac{f(q^n a) - f(q^{n+1}a)}{q^n a - q^{n+1}a} g(aq^n).$$

Notice that

$$\begin{aligned}\sum_{n=0}^{m} q^n \frac{f(q^n a) - f(q^{n+1}a)}{q^n a - q^{n+1}a} g(aq^n) &= \sum_{n=0}^{m} \frac{f(aq^n)g(aq^n) - f(aq^{n+1})g(aq^n)}{a(1-q)} \\ &= \frac{1}{a(1-q)}\left[\sum_{n=0}^{m} g(aq^n)(f(aq^n) - f(aq^{n+1}))\right].\end{aligned}$$

But then

$$\begin{aligned}\sum_{n=0}^{m} g(aq^n)(f(aq^n) - f(aq^{n+1})) &= f(a)g(aq^{-1}) - f(aq^{m+1})g(aq^m) + \sum_{n=0}^{m} f(aq^n)(g(aq^n) - g(aq^{n-1})) \\ &= f(a)g(aq^{-1}) - f(aq^{m+1})g(aq^m) + \\ &\qquad \sum_{n=0}^{m} f(aq^n)\frac{g(aq^n) - g(aq^{n-1})}{aq^n - aq^{n-1}}(aq^n - aq^{n-1}).\end{aligned}$$

Now letting $m \to \infty$ gives us

$$\begin{aligned}\int_0^a (D_q f) g \, d_q x &= f(a)g(aq^{-1}) - f(0)g(0) + a(1-q)\frac{1-q^{-1}}{1-q}\sum_{n=0}^{\infty}\left[q^n f(aq^n)\frac{g(aq^n) - g(aq^{n-1})}{aq^n - aq^{n-1}}\right] \\ &= f(a)g(aq^{-1}) - f(0)g(0) + \frac{q-1}{q}\cdot\frac{1}{1-q}\left[a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n) D_{q^{-1}} g(aq^n)\right] \\ &= f(a)g(aq^{-1}) - f(0)g(0) - \frac{1}{q}\int_0^a f(x) D_{q^{-1}} g(x) \, d_q x.\end{aligned}$$

□

Now that we introduced the q-integral, we are ready to state $\Gamma_q(z)$ in terms of a q-integral just like how the regular gamma function can be written in terms of an integral.

Proposition 6.1. *The q-gamma function $\Gamma_q(z)$ is*

$$\Gamma_q(z) := \int_0^{\frac{1}{1-q}} t^{z-1} E_q(-q(1-q)t) \, d_q t$$

for all $\mathrm{Re}(z) > 0$. $E(x)$ *is the* exponential function *defined by*

$$E_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} q^{\binom{n}{2}} = (-x;q)_\infty.$$

Proof. We have

$$\begin{aligned}\int_0^{\frac{1}{1-q}} t^{z-1} E_q(-q(1-q)t)\, d_q t &= \frac{1}{1-q}(1-q) \sum_{n=0}^{\infty} q^n \left(\frac{q^n}{1-q}\right)^{z-1} (q^{n+1};q)_\infty \\ &= (1-q)^{1-z} \sum_{n=0}^{\infty} q^{nz} \frac{(q;q)_\infty}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= (1-q)^{1-z} (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{nz}}{(q;q)_n} = \frac{(1-q)^{1-z}(q;q)_\infty}{(q^z;q)_\infty}.\end{aligned}$$

□

6.4. q-Beta function

Definition 6.4. The *beta function* $\mathrm{B}(x,y)$ is defined by

$$\mathrm{B}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt,$$

which converges for all $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

On $\mathrm{B}(x,y)$, perform the change of variables via

$$t = \frac{u}{1+u}.$$

The change of variable from t to u gives us

$$\mathrm{B}(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x-1}} \frac{1}{(1+u)^{y-1}} \frac{du}{(1+u)^2} = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Unfortunately, we cannot change variables in q-integrals. Nonetheless, the q-analogue of the beta function can be defined, and enjoys the analogous properties enjoyed by the regular beta function.

Definition 6.5. The *q-beta function* is defined by

$$\mathrm{B}_q(x,y) := \int_0^1 u^{x-1} \frac{(qu;q)_\infty}{(q^y u;q)_\infty}\, d_q u.$$

Theorem 6.6. *The q-beta function* $\mathrm{B}_q(x,y)$ *satisfies the following properties.*

(1) $\mathrm{B}_q(x,y) = \dfrac{1-q^{y-1}}{1-q^x}\mathrm{B}_q(x+1,y-1)$.
(2) $\mathrm{B}_q(x,y) = \mathrm{B}_q(x,y+1) + q^y \mathrm{B}_q(x+1,y)$
(3) $\mathrm{B}_q(x,y) = \dfrac{1-q^{x+y}}{1-q^y}\mathrm{B}_q(x,y+1) = \dfrac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$.

Proof. For the first claim, we have by integration by parts,

$$\frac{1-q^x}{1-q}\mathrm{B}_q(x,y) = \int_0^1 \frac{1-q^x}{1-q} u^{x-1} \frac{(qu;q)_\infty}{(uq^y;q)_\infty}\, d_q u$$

$$= -\frac{1}{q}\int_0^1 u^x \left(D_{q^{-1}} \frac{(qu;q)_\infty}{(uq^y;q)_\infty} \right) d_q u,$$

since the boundary terms add up to 0. Evaluating the $D_{q^{-1}}$ operator gives

$$\begin{aligned} D_{q^{-1}} \frac{(qu;q)_\infty}{(uq^y;q)_\infty} &= \left[\frac{(qu;q)_\infty}{(uq^y;q)_\infty} - \frac{(u;q)_\infty}{(uq^{y-1};q)_\infty} \right] \frac{1}{u(1-q^{-1})} \\ &= \frac{(qu;q)_\infty}{(uq^{y-1};q)_\infty} \frac{1-uq^{y-1}-(1-u)}{u(1-q^{-1})} = \frac{(qu;q)_\infty}{(uq^{y-1};q)_\infty} \frac{u(1-q^{y-1})}{u(1-q^{-1})}. \end{aligned}$$

Hence

$$\frac{1-q^x}{1-q} \mathrm{B}_q(x,y) = \frac{1-q^{y-1}}{1-q} \int_0^1 \frac{u^x (qu;q)_\infty}{(uq^{y-1};q)_\infty} d_q u,$$

and the claim follows upon noting that the integral portion is $\mathrm{B}_q(x+1, y-1)$.

For the second claim, note that

$$\begin{aligned} \mathrm{B}_q(x,y+1) &= \int_0^1 \frac{u^{x-1}(qu;q)_\infty}{(uq^{y+1};q)_\infty} d_q u = \int_0^1 u^{x-1} \frac{(qu;q)_\infty (1-uq^y)}{(uq^y;q)_\infty} d_q u \\ &= \int_0^1 u^{x-1} \frac{(qu;q)_\infty}{(uq^y;q)_\infty} - q^y \int_0^1 u^x \frac{(qu;q)_\infty}{(uq^y;q)_\infty} = \mathrm{B}_q(x,y) - q^y \mathrm{B}_q(x+1,y). \end{aligned}$$

The third one follows from the first two claims. From the first part, we have

$$\mathrm{B}_q(x,y+1) = \frac{1-q^y}{1-q^x} \mathrm{B}_q(x+1,y).$$

So by the second part,

$$\begin{aligned} \mathrm{B}_q(x,y) &= \mathrm{B}_q(x,y+1) + q^y \left(\frac{1-q^x}{1-q^y} \mathrm{B}_q(x,y+1) \right) \\ &= \frac{(1-q^y)+(q^y-q^{x+y})}{1-q^y} \mathrm{B}_q(x,y+1) = \frac{1-q^{x+y}}{1-q^y} \mathrm{B}_q(x,y+1). \end{aligned}$$

Now we have

$$\mathrm{B}(x,y) = \frac{1-q^{x+y}}{1-q^y} \mathrm{B}(x,y+1) = \cdots = \frac{(1-q^{x+y})(1-q^{x+y+1})\cdots(1-q^{x+y+n})}{(1-q^y)\cdots(1-q^{y+n})} \mathrm{B}(x,y+n+1).$$

But then

$$\begin{aligned} \lim_{n\to\infty} \mathrm{B}(x,y+n) &= \lim_{n\to\infty} \int_0^1 \frac{u^{x-1}(qu;q)_\infty}{(uq^{y+n};q)_\infty} d_q u \\ &= \int_0^1 u^{x-1}(qu;q)_\infty \, d_q u = (1-q) \sum_{n=0}^\infty \frac{q^n q^{n(x-1)} (q^{n+1};q)_\infty (q;q)_n}{(q;q)_n} \\ &= (1-q)(q;q)_\infty \sum_{n=0}^\infty \frac{q^n q^{nx} (q;q)_\infty}{(q;q)_n} = \frac{(1-q)(q;q)_\infty}{(q^x;q)_\infty}. \end{aligned}$$

So it follows that

$$\mathrm{B}_q(x,y) = \frac{(q^{x+y};q)_\infty (1-q)(q;q)_\infty}{(q^y;q)_\infty (q^x;q)_\infty} = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \qquad \square$$

6.5. **Evaluation of q-beta integrals**

Proposition 6.2. $\displaystyle\int_a^\infty f(x)\,dx = \int_a^{a+1}\left(\sum_{n=0}^{\infty} f(x+n)\right)dx.$

The observation above may appear pretty straightforward, but this evident observation led to some non-trivial results by Ramanujan.

Recall from Theorem 4.2 the Ramanujan ${}_1\psi_1$ sum is

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n = \sum_{n=-\infty}^{\infty}\frac{(bq^n;q)_\infty(a;q)_n z^n}{(b;q)_n(bq^n;q)_\infty} = (*).$$

Factoring a out in $(a;q)_n$ gives

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) = (-1)^n a^n q^{\binom{n}{2}}(q^{1-n}/a;q)_n,$$

so it follows

$$\begin{aligned}(*) &= \frac{1}{(b;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(bq^n;q)_\infty(q^{1-n}/a;q)_n q^{\binom{n}{2}}(-1)^n(a^{-1};q)_\infty}{(a^{-1};q)_\infty}\\ &= \frac{1}{(b,a^{-1};q)_\infty}\sum_{n=-\infty}^{\infty}(bq^n;q)_\infty\left(\frac{q}{aq^n};q\right)_\infty q^{\binom{n}{2}}e^{in\pi}(az)^n.\end{aligned}$$

Define

$$\begin{aligned}I(a,b) &:= \int_{\mathbb{R}}(bq^x,q^{1-x}/a;q)_\infty q^{\binom{x}{2}}(az)^x e^{i\pi x}w(x)\,dx = \int_0^1\sum_{n=-\infty}^{\infty}f(x+n)\,dx\\ &= \int_0^1\sum_{n=-\infty}^{\infty}(bq^{x+n},q^{1-n-x};q)_\infty q^{\binom{x+n}{2}}(az)^{x+n}e^{i\pi(x+n)}w(x+n)\,dx\\ &= \int_0^1\sum_{n=-\infty}^{\infty}(bq^{x+n},q^{1-n-x};q)_\infty q^{\binom{x+n}{2}}(az)^{x+n}e^{i\pi x}(-1)^n w(x+n)\,dx,\end{aligned}$$

where $w(x)$ has period 1. Now consider

$$\begin{aligned}(bq^{x+n},q^{1-n-x}/a;q)_\infty &= (bq^x,q^{1-x}/a;q)_\infty\frac{(q^{1-x-n}/a;q)_n}{(bq^x;q)_n}\\ &= \frac{(bq^x,q^{1-x}/a;q)_\infty}{(bq^x;q)_n}(-1)^n(aq^x;q)_n(-q^{1-x}/a)^n q^{-\binom{n+1}{2}},\end{aligned}$$

where the last step follows by applying the identity

$$(aq^{-n};q)_n = (qa^{-1};q)_n(-a)^n q^{-\binom{n+1}{2}}$$

onto $(q^{1-x-n}/a;q)_n$. So putting these together,

$$I(a,b) = \int_0^1(bq^x,q^{1-x}/a;q)_\infty q^{\binom{x}{2}}e^{i\pi x}(az)^x w(x)\sum_{n=-\infty}^{\infty}\frac{(aq^x;q)_n}{(bq^x;q)_n}z^n\,dx,$$

as long as $|b/a| < |z| < 1$. Now apply Theorem 4.2 onto the infinite sum to get

$$\begin{aligned} I(a,b) &= \frac{(b/a,q;q)_\infty}{(b/(az),z;q)_\infty}\int_0^1 w(x)\left(bq^x,\frac{q^{1-x}}{a};q\right)_\infty q^{\binom{x}{2}}e^{i\pi x}(az)^x\frac{(q^{1-x}/(az),aq^xz;q)_\infty}{(bq^x,q^{1-x}/a;q)_\infty}\,dx \\ &= \frac{(b/a,q;q)_\infty}{(b/(az),z;q)_\infty}\int_0^1 w(x)q^{\binom{x}{2}}e^{i\pi x}(az)^x(q^{1-x}/(az),aq^xz;q)_\infty\,dx. \end{aligned}$$

Observe that $q^{\binom{x}{2}}(az)^x e^{i\pi x}(azq^x,q^{1-x}/(az);q)_\infty$ has period 1. With this in mind, we choose $w(x)$ to be

$$w(x) = \frac{q^{-\frac{x}{2}}(az)^{-x}e^{-i\pi x}}{(azq^x,q^{1-x}/(az);q)_\infty}p(x),$$

where $p(x)$ is any unit-periodic function, i.e., $p(x+1)=p(x)$ such that $\displaystyle\int_0^1 p(x)\,dx \neq 0$. We also verify that $w(x)$ has period 1. Indeed as we desired, we have

$$\begin{aligned} \frac{w(x+1)}{w(x)} &= \frac{q^{\binom{x}{2}-\binom{x+1}{2}}}{az}(-1)\frac{(q^{1-x}/(az),azq^x;q)_\infty}{(q^{-x}/(az),azq^{x+1};q)_\infty} \\ &= \frac{q^{-x}}{az}(-1)\frac{1-azq^x}{1-q^{-x}/(az)} = 1. \end{aligned}$$

Therefore,

$$I(a,b) = \frac{(b/a,q;q)_\infty}{(b/(az),z;q)_\infty}\int_0^1 p(x)\,dx.$$

Finally, letting $q^x =: u$ (hence $x = \frac{\log u}{\log q}$) gives us

$$\int_0^\infty \frac{(bu,q/(au);q)_\infty}{(q/(auz),auz;q)_\infty}p\left(\frac{\log u}{\log q}\right)\frac{du}{u} = \frac{(b/a,q;q)_\infty}{(b/(az),z;q)_\infty}(-\log q)\int_0^1 p(x)\,dx.$$

A corollary is another Ramanujan-type integral: letting $p(x)\equiv 1$ and $(a,b,z) := (-q^{-\alpha},-q^{\beta},-q^{\alpha})$ gives us

$$\int_0^\infty \frac{(-tq^\beta,-q^{\alpha+1}/t;q)_\infty}{(-t,q/t;q)_\infty}\frac{dt}{t} = \frac{\log q}{1-q}\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

for all $0<q<1$ and $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$. Thus we have proved the following ideas.

Theorem 6.7. *We have the Ramanujan-type integration*

$$\int_0^\infty \frac{(bu,q/(au);q)_\infty}{(azu,q/(azu);q)_\infty}p\left(\frac{\log u}{\log q}\right)\frac{du}{u} = -\log q\frac{(q,b/a;q)_\infty}{(z,b/(az);q)_\infty}\int_0^1 p(x)\,dx.$$

Theorem 6.8 (q-analogue of the beta integral)**.** *We have the Ramanujan-type integral*

$$\int_0^\infty \frac{(-tq^\beta,-q^{\alpha+1};q)_\infty}{(-t,-q/t;q)_\infty}\frac{dt}{t} = \frac{\log q}{1-q}\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

whenever $0<q<1$ *and* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$.

7. q-Hermite polynomials

Definition 7.1. *Hermite polynomials* are orthogonal polynomials with respect to the normal distribution, i.e.,

$$\int_{\mathbb{R}} e^{-x^2} H_m(x) H_n(x)\, dx = \sqrt{\pi} 2^n n! \delta_{m,n}$$

defined by the relation

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x).$$

The *q-Hermite polynomials* are the q-analogues of Hermite polynomials, and is defined by the following recurrence relation.

$$2xH_n(x\,|\,q) = H_{n+1}(x\,|\,q) + (1-q^n)H_{n-1}(x\,|\,q),\ H_0(x\,|\,q) = 1, H_1(x\,|\,q) = 2x.$$

The closed form of H_n is

$$H_n(x\,|\,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta}.$$

Dividing both sides of by $(q;q)_n$ $(n \geq 1)$, we get

$$2x\frac{H_n(x\,|\,q)}{(q;q)_n} = (1-q^{n+1})\frac{H_{n+1}(x\,|\,q)}{(q;q)_{n+1}} + \frac{(1-q^n)H_{n-1}(x\,|\,q)}{(1-q^n)(q;q)_{n-1}}. \tag{7.1}$$

Now let

$$F(t) = \sum_{n=0}^{\infty} \frac{H_n(x\,|\,q)t^n}{(q;q)_n}.$$

Multiply both sides of (7.1) by t^{n+1}, and add each term for all $n = 0, 1, 2, \ldots$. This gives us

$$\begin{aligned}
2tx(F(t)-1) &= F(t) - F(qt) - 1 - \frac{H_1}{1-q}t + 1 + \frac{H_1}{1-q}qt + t^2F(t)\\
2xtF(t) &= F(t) - F(qt) + t^2F(t)\\
F(t) &= \frac{F(qt)}{1-2xt+t^2} = \frac{F(qt)}{(1-te^{i\theta})(1-te^{-i\theta})} = \frac{1}{(te^{i\theta}, te^{-\theta}; q)_n}F(q^nt) \to \frac{1}{(te^{i\theta}, te^{-\theta}; q)_\infty}.
\end{aligned}$$

as $n \to \infty$.

$$\frac{1}{(te^{i\theta};q)_\infty}\frac{1}{(te^{-i\theta};q)_\infty} = \sum \frac{e^{ik\theta}}{(q;q)_k}\sum \frac{e^{-ij\theta}}{(q;q)_j}.$$

Thus

$$\frac{H_n(x\,|\,q)}{(q;q)_n} = \sum_{k+j=n} \frac{e^{ik\theta}e^{-ij\theta}}{(q;q)_j(q;q)_k} = \sum_{j=0}^{n} \frac{e^{i\theta(n-2j)}}{(q;q)_j(q;q)_{n-j}}.$$

Theorem 7.1. $w(x\,|\,q) = \dfrac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\sqrt{1-x^2}}$ *on* $[-1,1]$, *and*

$$\int_{-1}^{1} w(x)\, H_m(x\,|\,q)H_n(x\,|\,q) = \frac{2\pi}{(q;q)_\infty}(q;q)_n\delta_{m,n}.$$

Before getting into the proof, we first claim that by induction one can prove that $H_n(-x\,|\,q) = (-1)^nH_n(x\,|\,q)$. We also need the following lemma for orthogonality.

Lemma 7.1. $\displaystyle\int_0^\pi 2^{ij\theta}(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, d\theta = \frac{\pi(-1)^j}{(q;q)_\infty}(1+q^j)q^{\binom{j}{2}}.$

Proof. Recall that

$$\int_{-\pi}^{\pi} e^{in\theta}\, d\theta = \begin{cases} 2\pi & (n=0)\\ 0 & (n \neq 0).\end{cases}$$

Thus we have

$$\begin{aligned}
(e^{2i\theta}, e^{-2i\theta}; q)_\infty &= \frac{1-e^{2i\theta}}{(q;q)_\infty}(qe^{2i\theta}, e^{-2i\theta}, q; q)_\infty\\
&= \frac{1-e^{2i\theta}}{(q;q)_\infty}\sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}}(\sqrt{q}e^{2i\theta})^n(-1)^n.
\end{aligned}$$

Hence the integral becomes

$$\begin{aligned}
\int_0^\pi 2^{ij\theta}(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, d\theta &= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}}(-1)^n\int_0^\pi e^{ij\theta}(1-e^{2i\theta})e^{2in\theta}\, d\theta\\
&= \frac{\pi}{(q;q)_\infty}\sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}}(\delta_{j+n,0} - \delta_{1+j+n,0})(-1)^n\\
&= \frac{\pi}{(q;q)_\infty}\left\{(-1)^nq^{\binom{j}{2}} + (-1)^jq^{\binom{j+1}{2}}\right\}
\end{aligned}$$

$$= \frac{\pi}{(q;q)_\infty}(-1)^n q^{\binom{j}{2}}(1+q^j). \qquad \square$$

Proof of Theorem 7.1. Suppose that n and m have different parity. Then the integral becomes 0, since the weight function $w(x\,|\,q)$ is an even function.

$$\begin{aligned}(e^{2i\theta};q)_\infty(e^{-2i\theta};q)_\infty &= \prod_{n=1}^{\infty}(1-q^n e^{2i\theta})(1-q^n e^{-2i\theta})\\ &= \prod_{n=1}^{\infty}(1-2q^n(2x^2-1)+q^{2n})\end{aligned}$$

Thus we may assume that m and n have the same parity. Without loss of generality, suppose that $m = n+2k$ for some $k \in \mathbb{N}$. Now evaluate

$$\int_0^\pi H_m(x\,|\,q)(e^{2i\theta}, e^{-2i\theta};q)_\infty e^{i(m-2k)\theta}\,d\theta.$$

Since H_n only has non-zero coefficients for $e^{in\theta}$, $e^{i(n-2)\theta}$, $e^{i(n-4)\theta}$, and so forth, we have

$$\begin{aligned}\int_0^\pi H_m(x\,|\,q)(e^{2i\theta}, e^{-2i\theta};q)_\infty e^{i(m-2k)\theta}\,d\theta &= \sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}_q \int_0^\pi e^{i(m-2j)\theta}(e^{2i\theta}, e^{-2i\theta};q)_\infty e^{i(m-2k)\theta}\,d\theta\\ &= \sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}_q \int_0^\pi e^{i\theta(m-2j+m-2k)}(e^{2i\theta}, e^{-2i\theta};q)_\infty\,d\theta.\end{aligned}$$

By the lemma above, we have

$$\frac{\pi}{(q;q)_\infty}\sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}_q (-1)^{m-j-k}(1+q^{m-j-k})q^{\binom{m-j-k}{2}}.$$

Replacing m by $m-j$ gives

$$\begin{aligned}\frac{\pi}{(q;q)_\infty}(-1)^k\sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}_q (-1)^j(1+q^{j-k})q^{\binom{j-k}{2}} &= \frac{\pi}{(q;q)_\infty}\sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}_q (-1)^j(1+q^{j-k})q^{\frac{(j-k)(j-k-1)}{2}}\\ &= \frac{\pi(-1)^k}{(q;q)_\infty}q^{k(k+1)/2}\sum_{j=0}^{m}(-1)^j\begin{bmatrix} m\\ j\end{bmatrix}_q (1+q^{j-k})q^2q^{-kj}.\end{aligned}$$

Now replacing q^{-j} with x^k gives

$$\frac{\pi(-1)^k}{(q;q)_\infty}q^{\binom{k+1}{2}}\left[(q^{-k};q)_m + q^k(q^{1-k};q)_m\right].$$

Notice that above expression becomes 0 if $0<k<m$. If $k=0$, then the integral becomes $\dfrac{\pi}{(q;q)_\infty}(q;q)_m\delta_{k,0}$. If $k=m$, then we have

$$\frac{\pi}{(q;q)_\infty}\{(q;q)_m\delta_{k,0} + (-1)^m q^{\binom{m+1}{2}}(q^{-m};q)_m\delta_{k,m}\} = \frac{2\pi}{(q;q)_\infty}(q;q)_m$$

since $(-1)^m q^{\binom{m+1}{2}}(q^{-m};q)_m = (q;q)_m$. So all in all, if $k=0$ or $k=m$, we have

$$\frac{\pi}{(q;q)_\infty}(q;q)_m(\delta_{k,0}+\delta_{k,m}).$$

Putting the things together, it follows that

$$\int H_m H_m w\,dx = \frac{\pi}{(q;q)_\infty}(q;q)_m \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q (\delta_{k,0}+\delta_{k,m}) = 2\pi \frac{(q;q)_m}{(q;q)_\infty}. \qquad \square$$

Theorem 7.2 (Rogers)**.** *We have*

$$H_m(x\,|\,q)H_n(x\,|\,q) = \sum_{k=0}^{\min(m,n)} \frac{(q;q)_m(q;q)_n}{(q;q)_k(q;q)_{m-k}(q;q)_{n-k}} H_{m+n-2k}(x\,|\,q).$$

Proof. Suppose

$$H_m H_n = \sum a_{m,n,k} H_{m+n-2k}.$$

We observe a few properties that the coefficients $a_{m,n,k}$ satisfy. First, $a_{m,0,k} = a_{n,0,k} = \delta_{k,0}$. Also, we need $a_{m,n,0} = 1$. The leading term of $H_n(x\,|\,q)$ must have the coefficient 2^n due to the recurrence relation satisfied by the q-Hermite polynomials.

By the recurrence relation we have

$$\begin{aligned}&\sum a_{m+1,n,k} H_{m+1+n-2k} + (1-q^m)\sum a_{m-1,n,k} H_{m+n-1-2k} \\ &= \sum a_{m,n,k}[H_{m+n-2k+1} + (1-q^{m+n-2k})H_{m+n-2k-1}],\end{aligned}$$

so it follows

$$a_{m+1,n,k} + (1-q^m)a_{m-1,n,k-1} = a_{m,n,k} + (1-q^{m+n+2-2k})a_{m,n,k-1}.$$

Upon changing the variable from k to $k+1$, we have

$$a_{m+1,n,k+1} - a_{m,n,k+1} = a_{m,n,k+1} + (1-q^{m+n-2k})a_{m,n,k} - (1-q^m)a_{m-1,n,k}. \tag{7.2}$$

If $k=0$, then

$$\begin{aligned}a_{m+1,n,1} - a_{m,n,1} &= (1-q^{m+n})a_{m,n,0} - (1-q^m)a_{m-1,n,0} \\ a_{m+1,n,1} - a_{m,n,1} &= 1 - q^{m+n} - 1 + q^m = q^m(1-q^n).\end{aligned}$$

So telescoping gives us

$$\sum_{r=0}^{m} a_{r+1,n,1} - a_{r,n,1} = a_{m+1,n,1} - a_{0,n,1} = (1-q^n)\left(\frac{1-q^{m+1}}{1-q}\right).$$

Now, plugging in $k=1$ in (7.2); solving for $a_{m+1,n,2}$ (again, taking advantage of telescoping) gives

$$a_{m+1,n,2} - a_{0,n,2} = \frac{(1-q^n)(1-q^{n-1})(1-q^{m+1})(1-q^m)}{(1-q)(1-q^2)}.$$

Continuing with induction on k, we see that

$$a_{m,n,k} = \frac{(q;q)_m(q;q)_n}{(q;q)_{m-k}(q;q)_{n-k}(q;q)_k}. \qquad \square$$

Consider $f(x) := \sum a_n p_n(x)$, where $p_n(x)$ satisfies $\int p_m p_n w\,dx = \delta_{m,n}$. If

$$a_n := \int_{\mathbb{R}} f(x)p_n(x)w(x)\,dx,$$

then

$$f(x)=\sum_{n=0}^{\infty}p_n(x)\int_{\mathbb{R}}f(y)p_n(y)w(y)\,dy=\int_{\mathbb{R}}f(x)\left[\sum_{n=0}^{\infty}p_n(x)p_n(y)\right]w(y)\,dy.$$

Does this work? Not quite because $\sum p_n(x)p_n(y)w(y)$ diverges. However, the following works (note that we *cannot* bring in the limit as we just saw).

Definition 7.2. Suppose that $\{P_n(x)\}$ is a sequence of monic polynomials such that $\deg P_n(x)=n$, and that $\{P_n(x)\}$ satisfies the orthogonality relation

$$\int_{-\infty}^{\infty}P_m(x)P_n(x)\,d\mu(x)=\zeta_n\delta_{m,n},$$

where μ is the Lebesgue measure. The *Poisson kernel* of $\{P_n(x)\}$ is defined by

$$P_r(x,y)=\sum_{n=0}^{\infty}r^nP_n(x)P_n(y)/\zeta_n.$$

Theorem 7.3 (Poisson kernel formula)**.** $f(x)=\lim_{r\to 1^-}\int_{\mathbb{R}}f(x)\left(\sum_{n=0}^{\infty}r^np_n(x)p_n(y)\right)w(y)\,dy.$

7.1. **Poisson kernel for q-Hermite polynomials**

From the linearization formula, multiplying both sides by r^ms^n gives

$$\left(\sum_{m=0}^{\infty}\frac{H_m(x\,|\,q)r^m}{(q;q)_m}\right)\left(\sum_{n=0}^{\infty}\frac{H_n(x\,|\,q)s^n}{(q;q)_n}\right)=\sum_{k,m,n}\frac{r^ms^nH_{m+n-2k}(x\,|\,q)}{(q;q)_k(q;q)_{m-k}(q;q)_{n-k}}.$$

Note that the left-hand side is equal to $(re^{i\theta},re^{-i\theta},se^{i\theta},se^{-i\theta};q)_\infty$; as for the right-hand side, changing the variables ($m\to k+m$ and $n\to k+n$) gives us

$$\begin{aligned}\sum_{k,m,n}\frac{r^{m+k}s^{n+k}}{(q;q)_k}\frac{H_N(x\,|\,q)}{(q;q)_{m+n}}&=\sum_{k,N,n}\frac{(rs)^k}{(q;q)_k}\frac{H_N(x\,|\,q)}{(q;q)_{m+n}}=\sum_{n=0}^{N}\frac{r^{N-n}s^n(q;q)_{m+n}}{(q;q)_{N-n}(q;q)_n}\\&=\frac{1}{(rs;q)_\infty}\sum_N\frac{H_N(x\,|\,q)}{(q;q)_n}\sum_{n=0}^{N}r^{N-n}s^n\begin{bmatrix}N\\n\end{bmatrix}_q.\end{aligned}$$

where $N:=m+n$. Now change of variables from r to $ue^{i\phi}$ and s to $ue^{-i\phi}$ gives us

$$\begin{aligned}&\frac{1}{(rs;q)_\infty}\sum_N\frac{H_N(x\,|\,q)}{(q;q)_n}u^nH_n(\cos\theta\,|\,q)\\&=\frac{1}{(ue^{i(\phi+\theta)},ue^{i(\phi-\theta)},ue^{i(\theta-\phi)},ue^{-i(\theta+\phi)};q)_\infty}\end{aligned}$$

In conclusion,

$$\sum\frac{H_n(\cos\theta\,|\,q)H_n(\cos\theta\,|\,q)}{(q;q)_n}t^n=\frac{(t^2;q)_\infty}{(te^{i(\phi+\theta)},te^{i(\phi-\theta)},te^{i(\theta-\phi)},te^{-i(\theta+\phi)};q)_\infty}.$$

8. Proof of the Rogers–Ramanujan identities

8.1. **The sum side of Rogers–Ramanujan I**

First note that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

and that

$$\sum_{n=0}^{\infty} \frac{H_n(x\,|\,q^{-1})}{(q;q)_n} t^n (-1)^n = (te^{i\theta}, te^{-i\theta}; q)_\infty.$$

Thus putting them together we have

$$\begin{aligned} \frac{H_n(x\,|\,q^{-1})}{(q;q)_n} t^n (-1)^n q^{\binom{n}{2}} &= \sum_{n=0}^{\infty} \frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} e^{i\theta(n-2k)} \\ &= \sum_{n=0}^{\infty} \frac{(-t)^n}{(q;q)_n} q^{\binom{n}{2}} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta} \\ &= \sum_{n\geq k} \frac{(-t)^n q^{\binom{n}{2}+k^2-nk} e^{i(n-2k)\theta}}{(q;q)_k (q;q)_{n-k}}. \end{aligned}$$

Changing the variable n to $n+k$ gives

$$\sum_{k=0}^{\infty} \frac{(-t)^{n+k}}{(q;q)_k (q;q)_n} q^{\binom{n+k}{2}+k^2-(n+k)k} e^{i(n-k)\theta}.$$

Simplifying the exponent gives

$$\binom{n+k}{2} + k^2 - (n+k)k = \binom{n}{2} + \binom{k}{2}.$$

All in all,

$$\begin{aligned} \frac{H_n(x\,|\,q^{-1})}{(q;q)_n} t^n (-1)^n q^{\binom{n}{2}} &= \left(\sum_{k=0}^{\infty} \frac{(-t)^k}{(q;q)_k} q^{\binom{k}{2}} e^{-ik\theta} \right) \left(\sum_{n=0}^{\infty} \frac{(-t)^n}{(q;q)_n} q^{\binom{n}{2}} e^{in\theta} \right) \\ &= (te^{-i\theta}; q)_\infty (te^{i\theta}; q)_\infty. \end{aligned}$$

Now consider the following integral, which Ismail and Stanton came up with. Using this integral, we can not only prove the Rogers–Ramanujan identities, but a general family of such identities. Define

$$I(t) := \int_0^{\pi} (te^{i\theta}, te^{-i\theta}; q)_\infty (e^{2i\theta}, e^{-2i\theta}; q)_\infty \, d\theta.$$

Then we have

$$\begin{aligned} I(t) &= \frac{1}{2} \int_{-\pi}^{\pi} (te^{i\theta}, te^{-i\theta}; q)_\infty (e^{2i\theta}, e^{-2i\theta}; q)_\infty \, d\theta \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} \frac{H_n(x\,|\,q^{-1})}{(q;q)_n} (-1)^n q^{\binom{n}{2}} (e^{2i\theta}, e^{-2i\theta}; q)_\infty \, d\theta \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-t)^n}{(q;q)_n}q^{\binom{n}{2}}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{q^{-k(n-k)}(q;q)_n}{(q;q)_k(q;q)_{n-2k}}H_{n-2k}(x\,|\,q)(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{t^{2n}}{(q;q)_{2n}}q^{\binom{2n}{2}-n(2n-n)}\frac{2\pi}{(q;q)_\infty}\frac{(q;q)_{2n}}{(q;q)_n}\\
&= \frac{\pi}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{t^{2n}}{(q;q)_n}q^{n(2n-1)-n^2} = \frac{\pi}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{t^{2n}}{(q;q)_n}q^{n^2-n}.
\end{aligned}
$$

The last line is important in proving Rogers–Ramanujan, so we formally re-record the result.

$$
I(t) = \frac{\pi}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{t^{2n}}{(q;q)_n}q^{n^2-n} = \int_0^{\pi}(te^{i\theta},te^{-i\theta};q)_\infty(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta. \tag{8.1}
$$

8.2. **The product side of Rogers–Ramanujan I**

Theorem 4.3 implies that

$$
(q,-z\sqrt{q},-\sqrt{q}/z;q)_\infty = \sum_{n=-\infty}^{\infty}q^{\frac{n^2}{2}}z^n.
$$

Apply this to $I(-\sqrt{q})$.

$$
\begin{aligned}
I(-\sqrt{q}) &= \frac{1/2}{(q;q)_\infty}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}q^{n^2/2}e^{in\theta}(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta\\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{2n^2}\int_0^{\pi}e^{2in\theta}(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta,
\end{aligned}
$$

since only the terms with n even remain. Hence,

$$
\begin{aligned}
I(-\sqrt{q}) &= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{2n^2}\frac{\pi(-1)^n}{(q;q)_\infty}(1+q^n)q^{\binom{n}{2}} = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{2n^2}\frac{\pi(-1)^n}{(q;q)_\infty}(1+q^n)q^{\binom{n}{2}}\\
&= \frac{\pi}{(q;q)_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{5n^2-n}{2}}(1+q^n) = \frac{2\pi}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n(q^5)^{n^2/2}q^{-n/2}\\
&= \frac{2\pi}{(q;q)_\infty^2}(q^5,q^{5/2-1/2},q^{5/2+1/2};q^5)_\infty = \frac{2\pi}{(q;q)_\infty}\frac{(q^5,q^3,q^2;q^5)_\infty}{(q;q)_\infty} = \frac{2\pi}{(q;q)_\infty}\frac{1}{(q,q^4;q^5)_\infty}.
\end{aligned}
$$

Hence this proves Rogers–Ramanujan I.

8.3. **Proof of Rogers–Ramanujan II**

Rogers–Ramanujan II can be proved similarly, but this time use the integral

$$
\begin{aligned}
I(q) &= \int_0^{\pi}(qe^{i\theta},qe^{-i\theta};q)_\infty(1-e^{2i\theta})(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta\\
&= \frac{1}{2}\int_{-\pi}^{\pi}(qe^{i\theta},qe^{-i\theta};q)_\infty(1-e^{2i\theta})(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta\\
&= \frac{1}{2(q;q)_\infty^2}\int_{-\pi}^{\pi}(q,e^{i\theta},qe^{-i\theta};q)_\infty(q,qe^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta.
\end{aligned}
$$

Now we can apply the Jacobi triple product to the two infinite products from the right-hand side. Applying the Jacobi triple product gives us

$$\begin{aligned}
I(q) &= \frac{1/2}{(q;q)_\infty^2} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} \left(-\frac{1}{\sqrt{q}}\right)^n \sum_{m=-\infty}^{\infty} \frac{m^2}{q^2}(-\sqrt{q})^m \\
&= \frac{\frac{1}{2}(2\pi)}{(q;q)_\infty^2} \left\{ \sum_{m=-\infty}^{\infty} q^{2m^2+m+\frac{m^2}{2}+\frac{m}{2}}(-1)^m - q \sum_{m=-\infty}^{\infty} q^{\frac{(2m+1)^2}{2}-\frac{2m+1}{2}}(-1)^m q^{\frac{m^2+m}{2}} \right\} \\
&= \frac{\pi}{(q;q)_\infty^2} \left\{ \sum_{m=-\infty}^{\infty} q^{\frac{1}{2}m(5m+3)}(-1)^m - \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{5m^2+7m+m}{2}} \right\} \\
&= \frac{\pi}{(q;q)_\infty^2} \left\{ \sum_{m=-\infty}^{\infty} (q^5)^{\frac{m^2}{2}} q^{\frac{3m}{2}}(-1)^m + q \sum_{m=-\infty}^{\infty} (q^5)^{\frac{m^2}{2}} q^{\frac{7m}{2}}(-1)^m \right\} \\
&= \frac{\pi}{(q;q)_\infty^2} \left\{ (q^5, q^{-3/2+5/2}, q^{5/2+3/2}; q^5)\infty - (q^5, q^{7/2+5/2}, q^{5/2-7/2}; q^5)_\infty \right\} \\
&= \frac{\pi}{(q;q)_\infty^2} \left\{ (q^5, q^1, q^4; q^5)_\infty - q(1-q^{-1})(q^5, q^6, q^4; q^5)_\infty \right\} \\
&= \frac{\pi}{(q;q)_\infty^2} \left\{ (q^5, q^1, q^4; q^5)_\infty - (q^5, q, q^4; q^5)_\infty \right\} \\
&= \frac{2\pi}{(q;q)_\infty} \frac{1}{(q^2, q^3; q^5)_\infty}.
\end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty},$$

as required.

Consider the function

$$f(m,n) := q^{\binom{m}{2}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} q^{mn}.$$

Then one sees that

$$f(0,q) = \frac{1}{(q, q^4; q^5)_\infty} \text{ and } f(1,q) = \frac{1}{(q^2, q^3; q^5)_\infty}.$$

Theorem 8.1. $f(m+2,q) + f(m+1,q) = q^m f(m,q)$.

Proof.

$$\begin{aligned}
f(m+1,q) - q^m f(m,q) &= q^{\binom{m+1}{2}} \sum_{n=0}^{\infty} \frac{q^{n^2} + (m+1)n}{(q;q)_n} - q^{m+\binom{m}{2}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \\
&= q^{\binom{m+1}{2}} \sum_{n=1}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n}(q^n - 1) \\
&= q^{\binom{m+1}{2}} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2+m(n+1)}}{(q;q)_n}
\end{aligned}$$

$$= -q^{\binom{m+1}{2}} \sum_{n=0}^{\infty} \frac{q^{n^2+(m+2)n+(m+1)}}{(q;q)_n} = -f(m+2,q). \qquad \square$$

Theorem 8.2 (Garrett–Ismail–Stanton)**.** *We have*

$$q^{\binom{m}{2}} \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{(-1)^m a_m(q)}{(q,q^4;q^5)_\infty} - \frac{(-1)^m b_m(q)}{(q^2,q^3;q^5)_\infty},$$

where

$$a_m(q) := \sum_{j=0}^{\infty} q^{j^2+j} \begin{bmatrix} m-j-2 \\ j \end{bmatrix}_q$$

$$b_m(q) := \sum_{j=0}^{\infty} q^{j^2} \begin{bmatrix} m-j-1 \\ q \end{bmatrix}_q.$$

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