MATH 5370C: COMBINATORICS
(NON-SPECIALIST COMPREHENSIVE EXAM EDITION)

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ABSTRACT. This special "non-specialist comprehensive exam edition" version of MATH 5370 lecture notes only covers the lectures relevant to the combinatorics non-specialist comprehensive exam. Specifically, Chapter 10 (Combinatorial Designs; this chapter studies block designs and Latin squares) and the last bit regarding property P_k are not part of the syllabus, so any section covering Chapter 10 and the property P_k bit are removed in this version.

1. SECTION 3.1: PIGEONHOLE PRINCIPLE

Theorem 1.1 (Pigeonhole principle). *If $n + 1$ objects are distributed into n boxes, then at least one box contains two or more objects.*

Proof. Do we need one? ;) $\square$

Remark. Note that the pigeonhole principle is non-constructive: we don't know which box has more than one object.

Example. If you have 7 green socks and 3 yellow socks in the dryer, then if you remove 3 socks you must have a pair.

Example. If there are n couples, how many of the $2n$ people must you select to guarantee you get a couple? You need to choose $n + 1$.

We introduce two other related principles:

Theorem 1.2. *If n objects are placed into n boxes with no box empty, then every box contains exactly one object.*

Theorem 1.3. *If n objects are placed into n boxes with no box containing more than one, then every box contains exactly one object.*

We can also re-formulate the pigeonhole principle using functions:

Theorem 1.4. *In terms of function $f : X \to Y$ where X and Y are finite:*
- *if $|Y| < |X|$, then f is not injective (one-to-one)*
- *if $|Y| = |X|$ then f is bijective (i.e., f is one-to-one/injective iff f is onto/surjective)*

Proposition 1.1. *Given m integers $a_1, \ldots, a_m$, there exist k and l with $0 \le k \le l \le m$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is divisible by m.*

Proof. Consider $a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_m$. This list contains m numbers. If m divides any of these, we are done. Otherwise, this means these all leave a remainder when divided by m, in the set $\{1, \ldots, m-1\}$. By the pigeonhole principle, there are two numbers, say $a_1 + \cdots + a_k, a_1 + \cdots + a_l$ that leave the same remainder (without loss of generality, assume $k < l$). Thus $(a_1 + \cdots + a_l) - (a_1 + \cdots + a_k) = a_{k+1} + \cdots + a_l$ divides m. This completes the proof. □

Example. If we choose 101 members from $1, 2, \ldots, 200$, then we want to prove that there are two (distinct two numbers, not two same numbers) such that one divides the other. To do so, we need to be a little more creative when it comes to building the boxes. To do so, we will write each member as $2^k \cdot a$ where a is odd. By the pigeonhole principle, there must be two of the 101 members with the same odd part, say $2^k a$ and $2^l a$ for some odd a. If $k = l$ then they have to be the same, so without loss of generality assume $k < l$. Now it follows that $2^k a \mid 2^l a$, so we are done.

Theorem 1.5. *A rational number a/b has a decimal expansion that eventually repeats.*

Proof. Note that there are infinitely many pigeons (the numbers in the decimal expansion) while there are finitely many boxes (available remainders, i.e., $\{0, 1, \ldots, b-1\}$), so at some point the expansion has to repeat. □

2. SECTION 3.2: PIGEONHOLE PRINCIPLE (STRONG FORM)

Theorem 2.1 (Strong pigeonhole principle)**.** *Let $q_1, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - (n-1)$ objects are distributed into n boxes, then either:*

- *Box 1 contains at least q_1 objects; or*
- *Box 2 contains at least q_2 objects; or*
- *...*
- *Box n contains at least q_n objects.*

Proof. Suppose that none of the statement holds. Then for each $1 \le i \le n$, Box i contains at most $q_i - 1$ objects. This means that we distributed $q_1 + \cdots + q_n - n$ objects, but this is a contradiction (1 fewer than what we started with). □

Corollary 2.1. *Let n, r be positive integers. If $n(r-1) + 1$ objects are distributed into n boxes, at least one box has $r = \underbrace{\left\lceil \dfrac{n(r-1)+1}{n} \right\rceil}_{average}$ objects.*

Example. A basket of fruit is arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put into the basket to guarantee that either there is at least 7 apples, or 9 bananas, or 12 oranges? The answer is 7+9+12-(3-1)=26.

Example. Imagine a spinner with 200 divisions that has the inner wheel and the outer wheel. For the inner spinner, you can colour each red or blue. For the outer wheel, you can colour red or blue, but each must have 100. We claim that there is a rotation of the inner disc with at least 100 matches (with the outer wheel colouring).

Solution: If we fix the position of the outer disc, then there are 200 positions for the smaller disc. For each, count the number of matches. If each sector matches exactly 100 times, then the total number of matches is $200 \cdot 100 = 20000$. And there are 200 "boxes" (available positions). Thus, by Corollary 2.1 above, one of the rotations must have at least $\lceil 20000/200 \rceil = 100$ matches.

Theorem 2.2 (Erdős & Szekeres). *Every sequence $a_1, \ldots, a_{n^2+1}$ of n^2 real numbers contain either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.*

Proof. For $k = 1, \ldots, n^2 + 1$, let m_k be the length of a longest increasing subsequence that begins at a_k. If any $m_k \geq n + 1$, then we're done, so assume that is not. That is, assume that $1 \leq m_k \leq n$. Now by Theorem 2.1, there must be $n + 1$ of these that have the same length, i.e., $m_{k_1} = m_{k_2} = \cdots = m_{k_{n+1}}$ (without loss of generality, assume $1 \leq k_1 < k_2 < \cdots < k_{n+1} \leq n^2 + 1$). [ugh, re-read this proof and finish writing this up] $\square$

3. September 11: Introduction to graph theory

We start off with some examples of complete graphs:
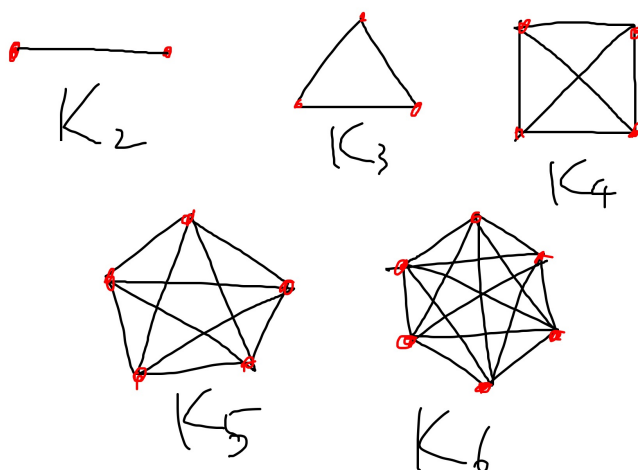
Can we colour the edges of K_4 so that there is no triangle whose edges are of the same colour? What about K_5 and K_6? Ramsey's theorem answers this question. In fact, for any K_n where $n \geq 6$, then this cannot be done.

Theorem 3.1 (Ramsey's theorem). *Of 6 or more people, either 3 mutual people are acquaintances, or 3 people are mutual strangers. In other words, no matter how edges of K_n are coloured (red or blue), there exists a red triangle or a blue triangle), for all $n \geq 6$.*

Theorem 3.2 (Ramsey's theorem, generalized). *If $m \geq 2$ and $n \geq 2$ are integers, there exists a positive integer p such that K_p can be broken down into K_m and K_n (i.e. $K_p \to K_m, K_n$), where K_m (resp. K_n) denotes the complete graph with m vertices coloured all red (resp. blue) edges.*

Definition 3.1. We define *Ramsey's number* $r(m, n)$ to be the smallest integer p that satisfies $K_p \to K_m, K_n$.

Remark. Some properties of Ramsey numbers:

- $r(m, n) = r(n, m)$
- $r(m, 2) = r(2, m) = m$
- $r(3, 3) = 6$.

Proof. First we will show that $r(m, n) \leq r(m - 1, n) + r(m, n - 1)$. That is, show that $K_{r(m-1,n)+r(m,n-1)} \to K_m, K_n$. Assume that $n, m \geq 3$. Suppose we have coloured edges of K_p where $p = r(m - 1, n) + r(m, n - 1)$, which we know exist by induction. Pick a vertex v_x and consider edges out of it to the rest. Put each remaining vertex into R_x if coloured red, and B_x if coloured blue. Then we have $|R_x| + |B_x| = p - 1 = r(m - 1, n) + r(m, n - 1) - 1$. By the pigeonhole principle, either $|R_x| \geq r(m - 1, n)$ or $|B_x| \geq r(m, n - 1)$.

(1) If $|R_x| \geq r(m - 1, n)$ then in R_x either there is a red K_{m-1} or there is a blue K_n. Note that, the red K_{m-1} together with v_x is a K_m.
(2) If $|B_x| \geq r(m, n - 1)$ then in B_x either there is a red K_m or there is a blue K_{n-1} (so together with v_x we have a blue K_n).

In either case we have $K_p \to K_m, K_n$, so such p exists and $p \leq r(m - 1, n) + r(m, n - 1)$. $\square$

Remark. There are extensions to more than two colours (e.g., $r(3, 3, 3) = 1$).

Can we prove by induction that

$$r(m, n) \leq \binom{m + n - 2}{m - 1}?$$

What do we know about $r(m, n)$?

Theorem 3.3 (Erdős). $r(n, n) > 2^{(n-1)/2}$ for all $n \geq 3$.

Proof. Take K_N where $N = \left\lfloor 2^{\frac{n-1}{2}} \right\rfloor$ and consider a random colouring of edges, red or blue. For $F \subseteq V(K_n)$, let E_S be event that S is either all red or all blue. Then the probability of E_s is

$$\Pr(E_s) = 2 \cdot \frac{1}{2^{\binom{n}{2}}}.$$

Thus the probability that some subset of n vertices is monochromatic is (call this event E_{mono})

$$\Pr(E_{\text{mono}}) = \Pr\left(\bigcup_S E_s\right) \leq \sum_S \Pr(E_s) = \binom{N}{n} 2 \cdot \frac{1}{2^{\binom{n}{2}}}$$
$$\leq \frac{N^n}{n!} 2 \cdot \frac{1}{2^{\binom{n}{2}}}$$
$$\leq \frac{2^{n(n-1)/2}}{n!} 2 \cdot \frac{1}{2^{\binom{n}{2}}} = \frac{2}{n!} < 1.$$

This implies that it is possible that all subsets of n vertices is not monochromatic. So there exists a 2-colouring of K_n with red or blue K_n. $\square$

4. September 11: Partial ordering and equivalence relations

Definition 4.1. A relation R on a set X is a *partial ordering* or a *partial order* on X if the following three conditions are satisfied:
   (i) (reflexivity) xRx for all $x \in X$
   (ii) (anti-symmetry) for any $x, y \in X$, if xRy and yRx, then $x = y$
   (iii) (transitivity) for all $x, y, z \in X$, if xRy and yRz, then xRz.

Definition 4.2. A relation R on X is said to be a *strict partial ordering* or a *strict partial order* if reflexivity in the above definition is replaced with irreflexivity, i.e., $x\cancel{R}x$ for all $x \in X$.

Example. $(X, \mathcal{P}(X))$ under $\subseteq$ is a partial ordering. If $\subseteq$ is replaced with $\subsetneq$, then it becomes to a strict partial ordering. $(\mathbb{N}, |)$ is a partial ordering, where $|$ denotes divisibility. $(\mathbb{R}, \leq)$ is a partial ordering, whereas $(\mathbb{R}, <)$ is a strict partial ordering.

Remark. We often denote a partial ordering (resp. a strict partial ordering) by "$\leq$" (resp. "$<$").

Remark. Given a partial order $\leq$, setting $x < y$ iff $x \leq y$ and $x \neq y$ yields a strict partial order. Similarly, given a strict partial order under $<$, setting $x \leq y$ iff $x < y$ or $x = y$ yields a partial order.

Definition 4.3. A set X on which a partial order $\leq$ is defined is called a *partially ordered set (poset)*. If R is a relation on X, then x and y are *comparable* if xRy or yRx; otherwise, they are *incomparable*. If every pair of a partial order R on set X is comparable, then R is said to be a *total ordering* or a *total order*.

Example. $(\mathbb{R}, \leq)$ is a total order, whereas $(\mathbb{N}, |)$ is not.

Theorem 4.1. *Let X be a finite set with n elements. Then there is a one-to-one correspondence between the total orders on X and the permutation of X. Therefore, the number of total orders on X is $n!$.*

First we prove the following lemma first.

Lemma 4.1. *Given a total order on X where X is a finite set with n elements, there exists a permutation $a_1, a_2, \ldots, a_n$ on X such that $a_1 < a_2 < \cdots < a_n$.*

Proof of Lemma 4.1. If $n = 1$, then this is trivial. Next, show that there exists a minimal element (which we can do because X is finite – this is why the finiteness condition is crucial here) so that $a_1 < x$ for all $x \in X \setminus \{a_1\}$. Repeat this operation, and the lemma follows by induction. $\square$

Proof of Theorem 4.1. Define $a_i \leq a_j$, provided that a_i comes before a_j in the permutation. This automatically makes $\leq$ a total order on R. This proves that we can construct a total order from the permutation. The other direction follows immediately from Lemma 4.1. $\square$

Definition 4.4. We say that x *is covered by* y or y *covers* x if $x < y$ and there is no z such that $x < z < y$. We write this as $x <_c y$. In other words, if a finite poset is represented geometrically (with a Hasse diagram), and if $x <_c y$, then x is below y, and there is a direct line form x to y.

Example. $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}, \{1,2,3\}\}$ with a partial order $\subseteq$. Then its Hasse diagram is
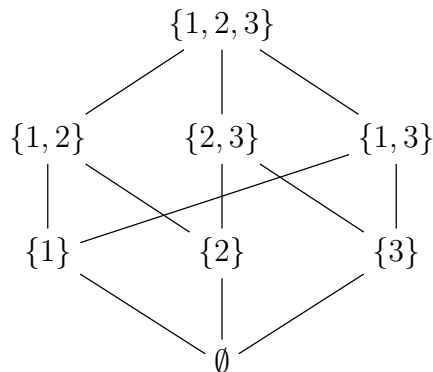
Definition 4.5. A *linear extension* $(X, \leq_2)$ of a poset $(X, \leq_1)$ is a poset such that $\leq_2 \supseteq \leq_1$ and $\leq_2$ is a total order.

Theorem 4.2. *Let $(X, \leq)$ be a finite poset. Then there is a linear extension of $(X, \leq)$.*

Example. $(\{1, \ldots, n\}, \leq)$ is a linear extension of $(\{1, \ldots, n\}, |)$.

5. September 13: Equivalence relation

Definition 5.1. Let X be a set. A relation R on X is an *equivalence relation* provided that it is:

- reflexive;
- symmetric: for all x and y, if xRy, then yRx; and
- transitive.

Equivalence relations are often written as $\sim$. We denote the equivalence class of a as $[a] = \{x : x \sim a\}$.

Remark. $\sim$ is the generalization of $=$.

Theorem 5.1. *Let $\sim$ be an equivalence relation on X. Then the distinct equivalence classes partition R into non-empty parts. Conversely, given any partition of X into nonempty parts, there is an equivalence relation on X whose equivalence classes are the parts of the partition.*

Proof. Suppose that $c \in [a] \cap [b]$. Then since $c \sim a$, via symmetry we have $a \sim c$. Thus we have $a \sim c$ and $c \sim b$, by transitivity $a \sim b$. Therefore $[a] = [b]$. Conversely, suppose that there is a partition of X. Then let $\sim$ be a relation such that $x \sim y$ whenever x and y belong to the same partition. It is a standard exercise to verify that $\sim$ is an equivalence relation. $\square$

Remark. Therefore, equivalence relations and partitions are two different ways of talking about the same concept.

Example. Let S_n be the set of all $n!$ permutations of $1, 2, \ldots, n$. Define R by $i_1, \ldots, i_n R j_1, \ldots, j_n$ whenever there is an integer k such that $j_1 \ldots j_n = i_k \cdots i_n i_1 \cdots i_{k-1}$. So we have $23451R45123$. Similarly we have $45123R34512$ and $23451R34512$. Thus $[23451] = \{34512, 45123, 51234, 12345\}$. In general, every class contains exactly n objects. So there are $n!/n = (n-1)!$ equivalence classes.

Example. Take all permutations of $1, 2, \ldots, n$. Define $i_1 \ldots i_k \sim j_1 \ldots j_k$ iff $\{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\}$. Then $[i_1 i_2 \ldots i_k] = \{$all permutation with the same elements $i_1, \ldots, i_k\}$. Note that the size of each equivalence class is $k!$. What about the number of equivalence classes? This is equal to the number of ways to choose k elements from $1, \ldots, n$. Thus there are $P(n, k)/k! = n!/((n-k)!k!) = \binom{n}{k}$ equivalence classes.

Example (Row reduction on $n \times m$ matrices). Let $A \sim B$ iff A is row-equivalent to B.

Example. On $\mathbb{Z}$, let $a \, R_n \, b$ if and only if $n \mid (a - b)$. The equivalent way of putting $a R_n b$ in this case is $a \equiv b \pmod{n}$ (the classes mod n).

6. SEPTEMBER 13: CHAPTER V - BINOMIAL COEFFICIENTS

Definition 6.1. Define the *binomial coefficient* $\binom{n}{k}$ as

$$
\binom{n}{k} := \begin{cases} 0 & \text{if } n < k; \\ 1 & \text{if } k = 0; \\ C(n,k) = \frac{n!}{k!(n-k)!} & \text{otherwise.} \end{cases}
$$

Proposition 6.1. $\binom{n}{k} = \binom{n}{n-k}$ *for any* $0 \le k \le n$.

Proposition 6.2 (Pascal's formula). $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

n \ k	0	1	2	3	4	$\cdots$
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	

The above is called Pascal's triangle, which illustrates Pascal's formula.

Proposition 6.3. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

Proof. The left-hand side denotes the number of ways to choose objects from n objects (how many ways can you choose 0 objects? what about 1? and so forth). On the other hand, the right-hand side denotes how many ways to choose objects from n objects also (For each object, you can choose or not choose - two options. There are n objects, so we have 2^n). $\square$

Definition 6.2. $\binom{n}{2} = \dfrac{n(n-1)}{2}$ is called a *triangular number*.

Theorem 6.1 (Binomial theorem). $(x + y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} y^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Corollary 6.1. $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{n-k} x^k.$

7. SEPTEMBER 18

Proposition 7.1. $k\binom{n}{k} = n\binom{n-1}{k-1}.$

Proof. Both describe how many ways one can choose a committee of k members with one president amongst n people. The LHS describes how many ways one can choose k committee members, and then pick a president from that. As for the RHS, this is equivalent to choosing a president first, and then choosing the remaining $k-1$ members from the remaining $n-1$ people. □

Proposition 7.2. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$ *for all* $n \geq 1$.

Proof. Let $x = -1$ where x is as it appears in Corollary 6.1. □

Proof. $\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$ for all $n \geq 1$. □

Proof. Differentiate both sides of Corollary 6.1, and then let $x = 1$. Another way of looking at this is as follows: the RHS depicts the total number of ways to pick a committee (choose a chairman from n people; now pick the remaining member of the committee from $n-1$ people); the same goes for LHS also (sum up how many ways you can choose a committee of 1 members, 2 members, etc). Note that there are $\binom{n}{k}$ ways to choose a committee of k members, and there are therefore k ways of picking a chairman. □

Proposition 7.3. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$

Proof. Each individual term of the LHS describes the following situation: choosing k people from the n people from the first den, and $n-k$ people for the n people in the second den (recall that $\binom{n}{k} = \binom{n}{n-k}$). This is equivalent to choosing how many ways you can choose n people amongst $2n$ people. □

8. SEPTEMBER 18: EXTENDED DEFINITION OF BINOMIAL COEFFICIENTS

Definition 8.1. Let r be any real and k any integer. Then

$$\binom{r}{k} := \begin{cases} \frac{r(r-1)(r-2)\cdots(n-k+1)}{k!} & (k \geq 1) \\ 1 & (k = 0) \\ 0 & (k \leq -1). \end{cases}$$

Example. $\binom{-7}{2} = \frac{(-7)(-7-1)}{2!} = 28.$

Example. $\binom{\frac{3}{2}}{3} = \frac{(-\frac{3}{2})(-\frac{3}{2}-1)(-\frac{3}{2}-2)}{3!} = -\frac{1}{16}.$

Proposition 8.1. *The following identities hold for any integer* k *and any real number* r:

(i) $\dbinom{r}{k} = \dbinom{r-1}{k} + \dbinom{r-1}{k-1}$

(ii) $k\dbinom{r}{k} = r\dbinom{r-1}{k-1}.$

Proposition 8.2. $\dbinom{r+k+1}{k} = \dbinom{r+k}{k} + \cdots + \dbinom{r+1}{1} + \dbinom{r}{0}.$

Proof. (i) can be proven via induction:

$$\dbinom{r}{k} = \dbinom{r-1}{k} + \dbinom{r-1}{k-1}$$
$$= \dbinom{r-1}{k} + \dbinom{r-2}{k-1} + \dbinom{r-2}{k-2}$$
$$\vdots$$
$$= \dbinom{r-1}{k} + \dbinom{r-2}{k-1} + \dbinom{r-3}{k-2} + \cdots + \dbinom{r-k}{1} + \dbinom{r-k-1}{0}.$$

Substituting $r+k+1$ for r yields the claim. $\qquad\square$

Proposition 8.3. *For all positive integers n and k,*

$$\dbinom{n+1}{k+1} = \dbinom{0}{k} + \dbinom{1}{k} + \cdots + \dbinom{n-1}{k} + \dbinom{n}{k}.$$

Corollary 8.1. $\dfrac{n(n+1)}{2} = \dbinom{n+1}{2} = \dbinom{0}{1} + \cdots + \dbinom{n}{1} = 1 + 2 + \cdots + n.$

9. September 18: Unimodality of binomial coefficients

Definition 9.1. The sequence $s_0, \ldots, s_n$ is *unimodal* if there is an integer t with $0 \le t \le n$ such that $s_0 \le s_1 \le \cdots \le s_t \ge s_{t+1} \ge \cdots \ge s_n$.

Example. $1, 4, 4, 2, 1$ is unimodal, but $1, 7, 8, 7, 9, 4$ is not.

Theorem 9.1. *Let n be a positive integer. Then $\dbinom{n}{0}, \dbinom{n}{1}, \ldots, \dbinom{n}{n}$ is unimodal. More-over, if n is even, then*

$$\dbinom{n}{0} < \dbinom{n}{1} < \dbinom{n}{2} < \cdots < \dbinom{n}{n/2} > \dbinom{n}{n/2+1} > \cdots > \dbinom{n}{n-1} > \dbinom{n}{n}.$$

If n is odd, then there is a double peak:

$$\dbinom{n}{0} < \dbinom{n}{1} < \cdots < \dbinom{n}{\frac{n-1}{2}} = \dbinom{n}{\frac{n+1}{2}} > \cdots > \dbinom{n}{n-1} > \dbinom{n}{n}.$$

Proof. Consider the following ratio:

$$\frac{\dbinom{n}{k}}{\dbinom{n}{k-1}} = \frac{n!}{k!(n-k)!} \cdot \frac{(k-1)!(n-k+1)!}{n!} = \frac{n-k+1}{k}.$$

Therefore,

$$\binom{n}{k-1} < \binom{n}{k} \Leftrightarrow n - k + 1 > k \Leftrightarrow k < \frac{n+1}{2}.$$

Similarly,

$$\binom{n}{k-1} = \binom{n}{k} \Leftrightarrow k = \frac{n+1}{2};$$

$$\binom{n}{k-1} > \binom{n}{k} \Leftrightarrow k > \frac{n+1}{2}.$$

So the equality (double peak) happens only when n is odd. $\qquad\square$

Corollary 9.1. *The largest of* $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ *(where n is a positive integer) is* $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$.

Definition 9.2. An *anti-chain* of a set S of size n is a collection $\mathfrak{A}$ of subsets of S such that no one is contained in another. Such collection is called a *Sperner family* or a *Sperner system.*

Remark. One can find anti-chains by choosing $k \in \{0, 1, \ldots, n\}$ and choose *all* subsets of S of size k. Then the size of this collection $\mathfrak{A}_k$ is $\binom{n}{k}$. Therefore $|\mathfrak{A}_k| \leq \max_k \binom{n}{k} = \binom{n}{\lfloor n/2 \rfloor}$.

10. September 20

Theorem 10.1. *Let S be a set of n elements. Then an anti-chain on S contains at most* $\binom{n}{\lfloor n/2 \rfloor}$ *sets.*

Definition 10.1. Define a *chain* $\mathfrak{C}$ of subsets: $A_1, A_2 \in \mathfrak{C} \Rightarrow A_1 \subset A_2$ or $A_2 \subset A$. A *maximal chain* is the one that cannot be made bigger.

Example. For $S = \{a, b, c, d, e\}$ and $\mathfrak{C} = \{\emptyset, \{a\}, \{a, c\}, \{a, c, d, e\}\}$, we see that $\mathfrak{C}$ is not maximal. However,

$$\{\emptyset, \{a\}, \{a, c\}, \{a, c, d\}, \{a, c, d, e\}, \{a, b, c, d, e\}\}$$

is maximal.

Remark. To construct a maximal chain $\mathfrak{C}$ of a set S (with $|S| = n$), order each element of S, and build a chain of subsets such that there is a one-to-one correspondence between $\mathfrak{C}$ and S. Therefore there are $n!$ maximal chains of S. Furthermore, any chain intersects any anti-chain in *at most* one set.

Proof of Theorem 10.1. Let $\mathfrak{A}$ be any anti-chain of S. Then define

$$\beta := \#\{(A, \mathfrak{C}) : A \in \mathfrak{A}, A \in \mathfrak{C}, \text{ and } \mathfrak{C} \text{ is a maximal chain of } S.\}.$$

We can count in two ways:

(1) $\beta \leq n!$ (At most one $A \in \mathfrak{A}$ is in an ordered pairs with a maximal chain $\mathfrak{C}$.)

(2) Define $\alpha_k := \#\{A \in \mathfrak{A} : \#A = k\}$. Then $\sum \alpha_k = |\mathfrak{A}|$. For a given $A \in \mathfrak{A}$ of size k, there are $k!(n-k)!$ maximal chains containing A.

Putting the two together, we see

$$\sum_{k=0}^{n} \alpha_k k!(n-k)! = \beta \leq n!$$

$$\sum_{k=0}^{n} \alpha_k \frac{k!(n-k)!}{n!} \leq 1$$

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1.$$

But then by unimodality, $\alpha_k / \binom{n}{k}$ becomes the smallest when $k = \lfloor n/2 \rfloor$. Hence it follows from the last inequality that

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{\lfloor n/2 \rfloor}} \leq 1.$$

Multiplying both sides by $\binom{n}{\lfloor n/2 \rfloor}$, and recognizing that $|\mathfrak{A}| = \sum_{k=0}^{n} \alpha_k$ yields the result. $\qquad \square$

Definition 10.2. If $n_1, \ldots, n_t$ are non-negative integers with $n = n_1 + \cdots + n_t$, then

$$\binom{n}{n_1, \ldots, n_t} := \frac{n!}{n_1! n_2! \ldots n_t!}$$

is called a *generalized binomial coefficient*, or a *multinomial coefficient*.

Theorem 10.2. $(x_1 + \cdots + x_t)^n = \displaystyle\sum_{n_1 + \cdots + n_t = n} \binom{n}{n_1, \ldots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.$

11. September 20: More on posets (Section 5.6)

Definition 11.1. In a poset $(X, \preceq)$, an *anti-chain* A is a subset of X such that no pair of A are comparable (i.e., $x, y \in A, x \neq y \Rightarrow x \not\preceq y, y \not\preceq x$) A *chain* is a subset of X such that any pair of elements are comparable (i.e., $x, y \in C \Rightarrow x \preceq y$ and $y \preceq x$).

Remark. If A is an anti-chain, and C is a chain, then $\#(A \cap C) \leq 1$.

Theorem 11.1. *Let $(X, \preceq)$ be a finite poset, and let r be the largest size of a chain. Then X can be partitioned into r, but no fewer anti-chains.*

Proof. Suppose $X_1 \preceq X_2 \preceq \cdots \preceq X_r$ is a chain of the largest size, and suppose that there are fewer than r anti-chains. But if this is the case, by the pigeonhole principle, there exists at least one anti-chain containing two or more chains, which is a contradiction. Thus there has to be at least r anti-chains.

Look at the maximal elements M_1 (those that have no element above them, i.e., $x \in M_1, y \neq x \Rightarrow x \not\preceq y$). Then M_1 is an anti-chain. Now, consider $(X - M_1, \preceq)$. The size of the largest chain in $(X - M_1, \preceq)$ is $r - 1$. Then let M_2 be the maximal elements in $(X - M_1, \preceq)$. Continue on and on until you reach M_r, which consists of the remaining elements after repeating this operation. $\qquad \square$

Theorem 11.2 (Dilworth's theorem). *Let $(X, \preceq)$ be a finite poset, and let m be the largest size of an anti-chain. Then X can be partitioned into m but no fewer chains.*

Proof. Read up on the proof! (Possibly a midterm question!) $\qquad\square$

12. September 25: Inclusion-exclusion principle (Chapter 6.1)

Theorem 12.1. *Let $P_1, \ldots, P_m$ be m properties referring to objects in S, and let*

$$A_i = \{x \mid x \in S \text{ has property } P_i\} \text{ for } i = 1, \ldots, m.$$

Then $|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|$. The alternate formulation is $|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| + \cdots + (-1)^{m-1} |A_1 \cap A_2 \cap \cdots \cap A_m|$.

Example. How many numbers between 1 and 1000 (inclusive) are *not* divisible by $5, 6$, and 8?
Solution: Let

$$A_1 := \{x \in \mathbb{N}_{1000} : 5 \,|\, x\} \quad A_2 \qquad := \{x \in \mathbb{N}_{1000} : 6 \,|\, x\} \quad A_3 := \{x \in \mathbb{N}_{1000} : 8 \,|\, x\}.$$

Note that $|A_1| = \lfloor 1000/5 \rfloor = 200, |A_2| = \lfloor 1000/6 \rfloor = 166, |A_3| = \lfloor 1000/8 \rfloor = 125, |A_1 \cap A_2| = \lfloor 1000/30 \rfloor = 33, |A_1 \cap A_3| = 25, |A_2 \cap A_3| = 41, |A_1 \cap A_2 \cap A_3| = \lfloor 1000/120 \rfloor = 8$. Therefore

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = 1000 - (200 + 166 + 126) + (33 + 25 + 41) - 8 = 600$$

out of 1000 numbers in $\{1, 2, \ldots, 1000\}$ are not divisible by 5, 6, and 8.

Example. How many permutations of $M, A, T, H, I, S, F, U, N$ are there so that none of the words MATH, IS, FUN appears?
Solution: There are total of 9! permutations. Now group each of the words, and treat it as one element. So there are 6! permutations containing MATH (permute MATH, I, S, F, U, N), 8! permutations containing IS, and 7! permutations containing FUN. Now, there are 5! permutations containing MATH and IS (MATH, IS, F, U, N); there are 4! permutations containing both MATH and FUN, and 6! permutations containing both IS and FUN. Finally, there are 3! ways to permute MATH, IS, FUN. Thus by inclusion-exclusion, we have

$$9! - (6! + 8! + 7!) + (5! + 4! + 6!) - 6 = 317658$$

permutations satisfying the desire property.

Example. How many integers between 0 and 99,999 (inclusive) have $2, 5$, and 8 amongst their digits?
Solution: Let

$$S := \{0, 1, \ldots, 99999\}$$
$$A_1 := \{x \in S : 2 \text{ does not appear as a digit in } X\}$$
$$A_2 := \{x \in S : 5 \text{ does not appear as a digit in } X\}$$
$$A_3 := \{x \in S : 8 \text{ does not appear as a digit in } X\}.$$

Then

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|$$

$$= 100000 - \binom{3}{1}9^5 + \binom{3}{2}8^5 - 7^5 = 4350,$$

so there are 4350 numbers that have 2, 5, and 8 amongst their digits.

13. September 25: Combinations with repetitions (Chapter 6.2)

Read this section yourself.

Example. How many 10-combinations of the multiset $T := \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$?

Example (Derangement). How many ways can you arrange n objects (say $1, 2, \ldots, n$) so that no object is in their correct/natural place? The following theorem answers this question.

Theorem 13.1. *Let D_n be the number of derangements of n objects. Then*

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^n\frac{1}{n!}\right).$$

Proof. Let S be the set of all permutations of $\{1, \ldots, n\}$. Let P_j be the property such that j is in its natural position. Let $A_j := \{x \in S : x \text{ has property } P_j\}$. Then

$$D_n = |\overline{A_1} \cap \cdots \overline{A_n}| = n! - \binom{n}{1}(n-1) + \binom{n}{2}(n-2)! + \cdots + (-1)^n\binom{n}{n}$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n\frac{1}{n!}$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^n\frac{1}{n!}\right). \qquad \square$$

Recall from calculus that

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots,$$

so

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots.$$

Therefore $e^{-1} \sim D_n/n!$, i.e., D_n is an integer closest to $n!/e$.

Proposition 13.1. $D_n = (n-1)(D_{n-2} + D_{n-1})$ *and* $D_n = nD^{n-1} + (-1)^n$.

Example. n men and n women bring coats to a dance. How many ways can the coats be returned with men getting men's coats, women getting women's coats, but no one gets the right coat?
Solution: No restrictions: $(2n)!$; men get men's coats, women get women's coats: $n! \times n!$; dearrangements: $D_n \times D_n$.

14. September 25 & 27: Recurrence relations and generating functions (Chapter 7)

Definition 14.1. A *recurrence relation* is one where current term (indexed by natural number n) is a function of the previous terms.

Example (Fibonacci sequence). Consider the sequence $(f_0, f_1, f_2, f_3, \dots) = (0, 1, 1, 2, 3, 5, \dots)$, i.e., $f_n = f_{n-1} + f_{n-2}$. Some interesting facts about Fibonacci:

(1) $s_n := f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$
(2) f_n is even if and only if $3 \mid n$.

Can we solve for f_n explicitly? We can try solution of the form $f_n = q^n$ for some non-zero q. Then $f_n = f_{n-1} + f_{n-2}$ becomes $q^n = q^{n-1} + q^{n-2}$, so $q^{n-2}(q^2 - q - 1) = 0$. But since $q^{n-2} \neq 0$, indeed $q^2 - q - 1 = 0$. So $q = \frac{1 \pm \sqrt{5}}{2}$. So

$$f_n = \left(\frac{1 \pm \sqrt{5}}{2} \right)^n$$

are both solutions. As recurrence is linear and homogeneous,

$$f_n = c_1 + \left(\frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

is also a solution, for some c_1 and c_2 – note that they are determined by base conditions. Then you will get $(c_1, c_2) = (\sqrt{5}^{-1}, -\sqrt{5}^{-1})$. Thus the generating function for f_n is

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n \qquad (n \geq 0).$$

Fibonacci numbers shows up pretty often in mathematics. More examples:

Example. Let h_n be the number of ways to perfectly cover a $2 \times n$ board with dominoes. Then $h_1 = 1, h_2 = 2, h_3 = 3$, and $h_n = h_{n-1} + h_{n-2}$.

Definition 14.2. Given a sequence $h_0, h_1, \dots, h_n, \dots,$ of numbers, its *generating function* is the infinite series

$$g(x) := h_0 + h_1 x + h_2 x^2 + \cdots .$$

So the coefficient of x^n in $g(x)$ is h_n.

Remark. If we have a finite sequence, then $g(x)$ is a polynomial.

Example. The generating function for $1, 1, 1, \dots$ is

$$g(x) = 1 + x + x^2 + \cdots = \frac{1}{1 - x}.$$

Example. The generating function for $\binom{m}{0}, \binom{m}{1}, \dots, \binom{m}{m}$ is

$$g(x) = \binom{m}{0} + \binom{m}{1} x + \cdots + \binom{m}{m} x^m = (1 + x)^m.$$

Example. Let K be an integer, and define sequence $h_0, \ldots, h_n, \ldots$ where

$$h_n := \binom{n+k-1}{k-1},$$

i.e., the number of non-negative integral solutions to $e_1 + \cdots + e_k = n$. Then the generating function is

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k} = (1 + x + x^2 + \cdots)^k = \sum x^{e_1} \sum x^{e_2} \cdots \sum x^{e_k}.$$

Example. Determine the generating function for the number of n-combinations of apples, bananas, oranges, and pears where, in each n-combination, the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and there is at least one pear.

Solution: The generating function is

$$g(x) = \underbrace{(1 + x^2 + x^4 + x^6 + \cdots)}_{\text{even number of apples}} \underbrace{(x + x^3 + x^5 + \cdots)}_{\text{odd number of bananas}} \underbrace{(1 + x + x^2 + x^3 + x^4)}_{\text{between 0 and 4 oranges}} \underbrace{(x + x^2 + x^3 + \cdots)}_{\text{at least one pear}}$$

$$= \frac{1}{1-x^2} \cdot \frac{x}{1-x^2} \cdot \frac{1-x^5}{1-x} \cdot \frac{x}{1-x} = \frac{x^2(1-x^5)}{(1-x)^2(1-x^2)^2}.$$

Can someone use generating functions to solve a counting problem?

Example. Find the number h_n of bags of fruits that can be made out of apples, bananas, oranges, and pears, where, in each bag, the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1.

Solution: Let $h(x)$ be the generating function, where

$$h(x) = h_0 + h_1 x + h_2 x^2 + \cdots$$
$$= (1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x + x^2 + x^3 + x^4)(1 + x)$$
$$= \frac{1}{1-x^2} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^5}{1-x} \cdot (1+x) = \frac{1}{1-x^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots.$$

Thus $h_n = n + 1$.

Example. Let g_n be the number of non-negative integral solutions to

$$3e_1 + 4e_2 + 2e_3 + 5e_4 = n.$$

Find a generating function $g(x)$ for $g_0, g_1, g_2, \ldots, g_n, \ldots$.
Solution: $g(x) = (1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots) = (1 - x^3)^{-1}(1 - x^4)^{-1}(1 - x^2)^{-1}(1 - x^5)^{-1}$.

Example. If you have an unlimited supply of pennies, nickels, dimes, quarters, and half-dollar coins, the generating function for f_n, the number of ways to make n cents with these coins, is $(1 - x)^{-1}(1 - x^5)^{-1}(1 - x^{10})^{-1}(1 - x^{25})^{-1}(1 - x^{50})^{-1}$.

15. September 27: Solving linear homogeneous recurrence relations (Chapter 7.4; skip Chapter 7.3)

Definition 15.1. Let $h_0, h_1, \ldots, h_n, \ldots$ be a sequence of members. We say that the sequence *satisfies a linear recurrence relations of order k* if there exist scalars $a_1, \ldots, a_k, b_n$ such that

$$h_n = a_1 h_{n-1} + \cdots + a_k h_{n-k} + b_n \qquad (n \geq k). \tag{1}$$

Particularly, if $b_n = 0$, then the given linear recurrence relations is *homogeneous.*

Example. Let D_n be the number of derangements of n objects. Then $D_n = (n-1)D_n + (n-1)D_{n-2}$, which is a homogeneous linear recurrence relation of order 2.

We will learn how to solve linear homogeneous recurrence relations (LHRR) with constant coefficients. Rewrite (1) as

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} = 0.$$

We start by looking at solutions ignoring the initial conditions.

Theorem 15.1. *Let q be a non-zero number. Then $h_n = q^n$ is a solution of the LHRR*

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} = 0 (a_k \neq 0, n \geq k)$$

with constant coefficients if and only if q is a root of the polynomial equation

$$x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_k = 0.$$

If the polynomial equation has k distinct roots $q_1, \ldots, q_k$ (which we shall call characteristic roots*), then*

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n \tag{2}$$

is the general solution of (1). Furthermore, no matter what initial values of $h_0, \ldots, h_{k-1}$ are given, there are $c_1, \ldots, c_k$ so that (2) is the unique *sequence that satisfies both the recurrence relation and the initial values.*

Proof. This is Theorem 7.4.1 of the text. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example. Solve $h_n = 2h_{n-1} + h_{h-2} - 2h_{h-3} (n \geq 3)$ with initial values $h_0 = 1, h_1 = 2, h_2 = 0$.
Solution: Characteristic equation is $x^3 - 2x^2 - x + 2 = 0$, and its roots are $q_1 = 1, q_2 = -1, q_3 = 2$. Thus the general solution is

$$h_n = c_1(1)^n + c_2(-1)^n + c_3(2)^n = c_1 + c_2(-1)^n + c_3 2^n.$$

Now solve for initial values, i.e.,

$$c_1 + c_2 + c_3 = 1 \quad (n = 0)$$
$$c_1 - c_2 + 2c_3 = 2 \quad (n = 1)$$
$$c_1 + c_2 + 4c_3 = 0 \quad (n = 2).$$

Solving this will yield the solution $(c_1, c_2, c_3) = (2, -\frac{2}{3}, -\frac{1}{3})$. So the solution is

$$h_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}2^n.$$

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Example. Words of length n using only the letters a, b, and c are to be transmitted with the condition that no word with two consecutive a's are to be transmitted. Determine the number of all words allowed.

Solution: Let h_n be the number of words allowed of length n. Then $h_0 = 1, h_1 = 3$ for $n \geq 2$. Then the LHRR is

$$h_n = 2h_{n-1} + 2h_{n-2}.$$

The first 2 comes covers the case when a word starts with a b or a c. The second 2 covers any words starting with a (since the second letter must be b or c). So the characteristic equation is $x^2 - 2x - 2 = 0$, so $x = 1 \pm \sqrt{3}$. Now we need to solve for c_1 and c_2, where

$$h_n = c_1(1 + \sqrt{3})^n + c_2(1 - \sqrt{3})^n.$$

Find c_1 and c_2 so that initial values $h_0 = 1$ and $h_1 = 3$ hold. In other words,

$$c_1 + c_2 = 1 \quad (n = 0)$$
$$c_1(1 + \sqrt{3}) + c_2(1 - \sqrt{3}) = 3 \quad (n = 1).$$

Solving this gives

$$(c_1, c_2) = \left(\frac{2 + \sqrt{3}}{2\sqrt{3}}, \frac{-2 + \sqrt{3}}{2\sqrt{3}} \right).$$

So the linear recurrence we are looking for is

$$h_n = \frac{2 + \sqrt{3}}{2\sqrt{3}}(1 + \sqrt{3})^n + \frac{-2 + \sqrt{3}}{2\sqrt{3}}(1 - \sqrt{3})^n.$$

In fact, it is possible to use generating functions to solve LHRR with constant coefficients.

Example. Solve $h_n = 5h_{n-1} - 6h_{n-2}$ where $n \geq 2$, with the initial values $h_0 = 1, h_1 = -2$.
Solution: Consider $g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots$. Then

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots$$
$$-5xg(x) = -5h_0 x - 5h_1 x^2 - \cdots - 5h_{n-1} x^n - \cdots$$
$$6x^2 g(x) = 6h_0 x^2 + \cdots + 6h_{n-2} x^n + \cdots.$$

Adding all three, we have

$$(1 - 5x + 6x^2)g(x) = h_0 + (h_1 - 5h_0)x + (h_2 - 5h_1 + 6h_0)x^2 + \cdots + (h_n - 5h_{n-1} + 6h_{n-2})x^n + \cdots.$$

Note $h_n - 5h_{n-1} + 6h_{n-2} = 0$ for all $n \geq 2$, and also $h_0 = 1$ and $h_1 = -2$, so

$$(1 - 5x + 6x^2)g(x) = h_0 + (h_1 - 5h_0)x = 1 - 7x.$$

Now use the partial fraction to get c_1 and c_2:

$$g(x) = \frac{1 - 7x}{1 - 5x + 6x^2} = \frac{1 - 7x}{(1 - 2x)(1 - 3x)} = \frac{c_1}{1 - 2x} + \frac{c_2}{1 - 3x}$$
$$1 - 7x = c_1(1 - 3x) + c_2(1 - 2x) = (c_1 + c_2) + (-3c_1 - 2c_2)x.$$

This gives us the system of equations: $c_1 + c_2 = 1, -3c_1 - 2c_2 = -7$. Thus $(c_1, c_2) = (5, -4)$. Hence,

$$g(x) = \frac{5}{1 - 2x} - \frac{4}{1 - 3x}.$$

Recall that

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots$$

$$\frac{1}{1-3x} = 1 + (3x) + (3x)^2 + (3x)^3 + \cdots .$$

Thus

$$g(x) = 1 + (-2)x + (-16)x^2 + \cdots + (5 \cdot 2^n - 4 \cdot 3^n)x^n + \cdots .$$

Thus $h_n = 5 \cdot 2^n - 4 \cdot 3^n$ for $n \geq 0$.

It is possible for some recurrences to have no general solution.

Example. The recurrence $h_n = 4h_{n-1} - 4h_{n-2}(n \geq 2)$ with $h_0 = 1, h_1 = 3$ has the characteristic equation $x^2 - 4x + 4 = (x-2)^2 = 0$, which has roots 2 and 2. Thus $h_n = c_1 2^n + c_2 2^n = C2^n$ for some C. Now, note that $h_0 = C = 1$ but $h_1 = 2C = 3$, but this is a contradiction.

15.1. More on the generating function method

Example. Let $h_0, h_1, \ldots, h_n$ be a sequence of numbers satisfying the recurrence

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0 \quad (n \geq 3)$$

where $h_0 = 0, h_1 = 1, h_2 = -1$. Solve this recurrence.

Solution: Let $g(x) = h_0 + h_1 x + h_2 x^2 + \cdots$. Then

$$g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots + h_n x^n + \cdots$$

$$xg(x) = h_0 x + h_1 x^2 + h_2 x^3 + \cdots + h_{h-1} x^n + \cdots$$

$$-16x^2 g(x) = -16h_0 x^2 - 16h_1 x^3 - \cdots - 16h_{n-2} x^n - \cdots$$

$$20x^3 g(x) = 20h_0 x^3 + \cdots + 20h_{n-3} x^n + \cdots .$$

Adding all four gives

$$(1 + x - 16x^2 + 20x^3)g(x) = h_0 + (h_1 + h_0)x^2 + (h_2 + h_1 - 16h_0)x^2 = x,$$

since $h_3 + h_2 - 16h_1 + 20h_0 = h_4 + h_3 - 16h_2 + 20h_1 = \cdots = h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0$. Hence,

$$g(x) = \frac{x}{1 + x - 16x^2 + 20x^3} = \frac{x}{(1-2x)^2(1+5x)} = \frac{c_1}{1-2x} + \frac{c_2}{(1-2x)^2} + \frac{c_3}{1+5x}$$

$$x = c_1(1-2x)(1+5x) + c_2(1+5x) + c_3(1-2x)^2$$

$$= (c_1 + c_2 + c_3) + (3c_1 + 5c_2 - 4c_3)x + (-10c_1 + 4c_3)x^2.$$

Hence we have the system of equations

$$c_1 + c_2 + c_3 = 0$$
$$3c_1 + 5c_2 - 4c_3 = 1$$
$$-10c_1 + 4c_3 = 0,$$

which yields $(c_1, c_2, c_3) = (-\frac{2}{49}, \frac{7}{49}, -\frac{5}{49})$. Hence

$$g(x) = -\frac{2}{49(1-2x)} + \frac{7}{49(1-2x)^2} - \frac{5}{49(1+5x)}$$

$$= -\frac{2}{49}\sum_{k=0}^{\infty} 2^k x^k + \frac{7}{49}\sum_{k=0}^{\infty}(k+1)2^k x^k - \frac{5}{49}\sum_{k=0}^{\infty}(-5)^k x^k$$

$$= \sum_{k=0}^{\infty}\left(\left(-\frac{2}{49}\right)2^k + \frac{7}{49}(k+1)2^k - \frac{5}{49}(-5)^k\right)x^k.$$

Therefore,

$$h_n = \frac{-2 \cdot 2^n + 7(n+1)2^n - 5(-5)^n}{49}$$

for all $n \geq 0$.

The general solution suggests that the roots of the characteristic equation are $2, 2, -5$. Let's see if this is the case. Recall that the characteristic equation is $r(x) = 0$ where $r(x) = x^3 + x^2 - 16x + 20$. Replacing x by x^{-1} and then multiplying by x^3 we get

$$x^3 r(x^{-1}) = 1 + x - 16x^2 + 20x^3 =: q(x),$$

which is the same as in $h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0$. And indeed, the roots of $r(x)$ are $2, 2, -5$. Since $r(x) = (x-2)^2(x+5)$ and $q(x) = x^3(x^{-1}-2)^2(x^{-1}+5) = (1-2x)^2(1+5x)$, which is the same as before.

Indeed, this can be done in general.

Theorem 15.2. *Let $h_0, h_1, \ldots, h_n, \ldots$ be a sequence of numbers that satisfy the LHRR*

$$h_n + c_1 h_{n-1} + c_2 h_{n-2} + \cdots + c_k h_{n-k} = 0 \quad (c_n \neq 0, n \geq k) \tag{3}$$

of order k with constant coefficients. Then its generating function $g(x)$ is of the form

$$g(x) = \frac{p(x)}{q(x)}, \tag{4}$$

where $\deg g = k$ and $\deg p < k$. Conversely, given such polynomials $p(x)$ and $q(x)$, there is a sequence $h_0, h_1, \ldots, h_n, \ldots$ satisfying a LHRR with constant coefficients of order k of the type (3) *whose generating function is given by* (4).

16. October 4

For repeated roots for the characteristic equation for a homogeneous linear recurrence, say q of multiplicity s, then

$$a^n, nq^n, \ldots, n^{s-1}q^n$$

will all be solutions, so we can take any linear combination of them.

Theorem 16.1. *Let $q_1, \ldots, q_n$ be the* distinct *roots of the characteristic equation of the LHRR $h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}$ $(a_k \neq 0, n \geq k)$. If q_i is an s_i-fold root of the characteristic equation, then the part of the general solution of this recurrence relation corresponding to q_i is $H_n^{(i)} = c_1 q_i^n + c_2 n q_i^n + \cdots + c_{s_i} n^{s_i-1} q^n$. Then the general solution of the recurrence is*

$$h_n = H_n^{(1)} + \cdots + H_n^{(t)}.$$

Example. Solve the recurrence relation

$$h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4} \quad (n \geq 4),$$

subject to $h_0 = 1, h_1 = 0, h_2 = 1, h_3 = 2$. Then $x^n = -x^{n-1} + 3x^{n-2} + 5x^n - 3 + 2x^{n-4}$, so the characteristic equation is $x^4 + x^3 - 3x^2 - 5x - 2 = 0$. By the rational root theorem, we see that $\pm 1, \pm 2$ are possible roots. Out of these four, we see that -1 is a root. Long division gives us the factorization $(x+1)(x+1)(x^2 - x - 2) = (x+1)^3(x-2) = 0$, so the roots are $-1, -1, -1, 2$. The general solution is

$$h_n = c_1(-1)^n + c_2 n(-1)^n + c_3 n^2(-1)^n + c_4 2^n.$$

Solve for c_1, c_2, c_3, c_4 with the given initial values:

$$c_1 + c_4 = 1$$
$$-c_1 - c_2 - c_3 + 2c_4 = 0$$
$$c_1 + 2c_2 + 4c_3 + 4c_4 = 1$$
$$-c_1 - 3c_2 - 9c_+ 8c_4 = 2.$$

This gives $(c_1, c_2, c_3, c_4) = (\frac{7}{9}, -\frac{1}{3}, 0, \frac{2}{9})$, so the general solution is

$$h_n = \frac{7}{9}(-1)^n - \frac{1}{3}n(-1)^n + \frac{2}{9}2^n.$$

So far we only have covered solving homogeneous ones. Now we will think about how to solve the non-homogeneous ones as well. First, we outline the steps to solve non-homogeneous recurrences.

(1) Find the general solution to the corresponding homogeneous recurrence relation.
(2) Find a particular solution of the non-homogeneous relation. This part is an art rather than a science. However, certain types of non-homogeneous part (say b_n) suggest trying certain particular solutions. For instance, if b_N is a polynomial of degree k in n, then try a polynomial of degree k in n. If b_n is an exponential of degree k in n, then try some exponential (e.g., if $b_n = d^n$ then try pd^n for some p).
(3) Sum up what you get in the first step and the second step, and then determine values of constants to satisfy the initial condition.

Example. Solve for $h_n = 2_{n-1} + 3^n$ for all $n \geq 1$ and $h_0 = 2$.
Solution: First, consider the corresponding homogeneous relation $h_n = 2h_{n-1}$ has the general solution $h_n = c_1 2^n$. Try $p \cdot 3^n$ as a particular solution (and search for p). Then $p \cdot 3^n = 2(p \cdot 3^{n-1}) + 3^n = (2p+1)3^n$, so $3p = 2p + 3$. Thus $p = 3$. So 3^{n+1} is a particular solution. Thus all solutions to the original recurrence are of the form $h_n = c_1 2^n + 3^{n+1}$. Plugging in $h = 0$ gives $c_1 = -1$. Verify that $h_n = -2^n + 3^{n+1}$ works by checking the recurrence relation and the initial condition – you will see this works.

Alternately, we can use the generating function to solve this.
Solution: Let $g(x) = h_0 + h_1 x + \cdots + h_n x^n + \cdots$. Then $(1 - 2x)g(x) = h_0 + (h_1 - 2h_0)x + (h_2 - 2h_1)x^2 + \cdots + (h_n - 2h_{n-1})x^n + \cdots$. Then $h_0 = 2$ and $h_n - 2h_{n-1} = 3^n$ for any $n \geq 1$. So we see that $(1 - 2x)g(x) = 2 + 3x + 3^2 x^2 + \cdots + 3^n x^n = 1 + 1 + (3x) + (3x)^2 + \cdots = 1 + (1 - 3x)^{-1}$. So $g(x) = (1 - 2x)^{-1} + (1 - 2x)^{-1}(1 - 3x)^{-1}$, and using the partial fraction method gives us

$$g(x) = \frac{1}{1 - 2x} + \frac{1}{(1 - 2x)(1 - 3x)} = \frac{1}{1 - 2x} - \frac{2}{1 - 2x} + \frac{3}{1 - 3x} = -\frac{1}{1 - 2x} + \frac{3}{1 - 3x}.$$

Thus

$$g(x) = \sum_{n=0}^{\infty}(2x)^n + (-2)\sum_{n=0}^{\infty}(2x)^n + 3\sum_{n=0}^{\infty}(3x)^n$$

$$= \sum 2^n x^n + \sum(-2)2^n x^n + \sum 3 \cdot 3^n x^n$$

$$= \sum_{n=0}^{\infty}(-2^n + 3^{n+1})x^n,$$

so $h_n = -2^n + 3^{n+1}$ as required, for all $n \geq 0$.

17. OCTOBER 11: SYSTEMS OF DISTINCT REPRESENTATIVES (CHAPTER 9)

Definition 17.1. Let Y be a finite set and $\mathcal{O} = (A_1, \ldots, A_n)$ be a family of subsets of Y (some may be the same). A *system of distinct representatives (SDR)* is a collection $(e_1, \ldots, e_n)$ of distinct elements of Y such that $e_1 \in A_1, \ldots, e_n \in A_n$.

Example. For $Y = \{a, b, c, d, e\}$, $A_1 = \{a, b, c\}$, $A_2 = \{b, d\}$, $A_3 = \{a, b, d\}$, $A_4 = \{b, d\}$, one can choose (c, b, a, d) to form a SDR. Another possible option: (c, d, a, b). However, if you choose a for A_1, then one cannot choose any representative for A_3.

Example. Let $Y = \{x, y, z\}$, $A_1 = \{x, y, z\}$, $A_2 = \{x, z\}$, $A_3 = \{x, y\}$, $A_4 = \{y, z\}$. In this case we cannot have any SDR – we only have three elements, but there are more than three sets.

Example. We want to place non-attacking rooks in a 4 board with forbidden squares. If $A_1 = \{1, 3, 4, 5\}$, $A_2 = \{1, 2, 4, 5\}$, $A_3 = \{2, 4\}$, and $A_4 = \{2, 3, 4, 5\}$, then one can place four non-attacking rooks on board i.e., there is an SDR for A_1, A_2, A_3, A_4.

Consider the following general problem. Let $\mathcal{O} = (A_1, \ldots, A_n)$ is a family of subsets of a finite set Y. Then determine if $\mathcal{O}$ has an SDR; if not what is the largest number t of sets in a subfamily $\mathcal{O}(i_1, \cdots i_t) := (A_{i_1}, A_{i_2}, \ldots, A_{i_t})$ that does have an SDR. It is not that hard to come up with a necessary condition: $\left| \bigcup_{t=1}^{k} A_{i_t} \right| \geq k$. Note that you need to have enough elements in order to pick a representative for each of the k sets. For this, you want to have at least k distinct elements to guarantee a pick for each set. This is also known as the "marriage condition". But surprisingly, this is also a sufficient condition, as we will see in the theorem below.

Theorem 17.1. *The family $\mathcal{O} = (A_1, \ldots, A_n)$ of subsets of a finite set Y has an SDR if and only if for all $k \in \{1, 2, \ldots n\}$ and for all $A_{i_1}, \ldots, A_{i_k}$ we have $|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \geq k$.*

Proof. ($\Rightarrow$) already proved.
($\Leftarrow$) We will prove this by induction.

(i) Base case ($n = 1$): By the marriage condition, indeed $|A_1| \geq 1$, so just pick $x_1 \in A_1$.
(ii) Inductive hypothesis: Assume that if MC holds, then there exists an SDR for any collection containing fewer than n sets.

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(iii-1) Inductive step - Case 1 ("tight case"): for some $k \in \{1, \ldots, n-1\}$, there is a subfamily $A_{i_1}, \ldots, A_{i_k}$ such that $|A_{i_1} \cup \cdots \cup A_{i_k}| = k$. Without loss of generality, re-index each of A_{i_t} with A_t. Write $E := A_1 \cup \cdots \cup A_k$. Then $|E| = k$. As $\mathcal{O}$ satisfies the MC, so does $\mathcal{O}' = (A_1, \ldots, A_k)$. So by inductive hypothesis $\mathcal{O}'$ has an SDR (say $(e_1, \ldots, e_k)$). Now set $\mathcal{O}^* := \{A_{k+1} \setminus E, \ldots, A_n \setminus E\}$. Clearly $\mathcal{O}^*$ has fewer than n sets. We want $\mathcal{O}^*$ to have the marriage condition. Pick any l sets, say $A_{j_1} \setminus E, A_{j_2} \setminus E, \ldots, A_{j_l} \setminus E$. Now consider the $k + l$ sets $A_1, \ldots, A_k, A_{j_1}, \ldots, A_{j_l}$. Then $|A_1 \cup \cdots \cup A_k \cup A_{j_1} \cup \cdots \cup A_{j_l}| \geq k + l$. Then $|E \cup A_{j_1} \cup \cdots \cup A_{j_l}| \geq k + l$. But since E is already in the union, it follows

$$|E \cup (A_{j_1} \setminus E) \cup \cdots \cup (A_{j_l} \setminus E)| \geq k + l.$$

But $E \cap (A_{j_t} \setminus E) = \emptyset$ for each t, so indeed

$$\underbrace{|E|}_{k} + |(A_{j_1} \setminus E) \cup \cdots \cup (A_{j_l} \setminus E)| \geq k + l,$$

or equivalently

$$|(A_{j_1} \setminus E) \cup \cdots \cup (A_{j_l} \setminus E)| \geq l.$$

Hence $\mathcal{O}^*$ satisfies the marriage condition, so by induction it has an SDR $(e_{k+1}, \ldots, e_n)$. Putting the two pieces together, we see that $(e_1, \ldots, e_n)$ is an SDR for $\mathcal{O}$.

(iii-2) Inductive step - Case 2: For all $k \in \{1, \ldots, n-1\}$ and all indices $i_1, \ldots, i_k$, we have $|A_{i_1} \cup \cdots \cup A_{i_k}| > k$. So $|A_1| \geq 2$, so pick $e_1 \in A_1$. And now consider $\mathcal{O}' = (A_2 \setminus \{e_1\}, \ldots, A_n \setminus \{e_1\})$. Note that $\mathcal{O}'$ contains fewer than n sets, so we are home free if $\mathcal{O}'$ has the marriage condition. Choose some k sets from $\mathcal{O}'$, say $|(A_{i_1} \setminus \{e_1\}) \cup \cdots \cup (A_{i_k} \setminus \{e_1\})| = |(A_{i_1} \cup \cdots \cup A_{i_k}) \setminus \{e_1\}|$. Since $|(A_{i_1} \cup \cdots \cup A_{i_k}) \setminus \{e_1\}| \geq |A_{i_1} \cup \cdots \cup A_{i_k}| - 1 \geq k$, by induction $\mathcal{O}'$ has an SDR $(e_2, \ldots, e_n)$. Then $(e_1, \ldots, e_n)$ is an SDR for $\mathcal{O}$. $\square$

Corollary 17.1. *Let $\mathcal{O} = (A_1, \ldots, A_n)$ be a family of subsets of a finite set Y. Then the largest number of sets in a subfamily of $\mathcal{O}$ with an SDR equals the smallest value of*

$$|A_{i_1} \cup \cdots \cup A_{i_k}| + n - k$$

over all choices of $k = 1, \ldots, n$ and all choices of k indices $i_1, \ldots, i_k$.

Example. Let $A_1 = \{a, b, c\}$, $A_2 = \{b, c\}$, $A_3 = \{b, c\}$, $A_4 = \{b, c\}$, $A_5 = \{c\}$, $A_6 = \{a, b, c, d\}$. Clearly we cannot find an SDR for all six sets. However, note that $|A_2 \cup A_3 \cup A_4 \cup A_5| = |\{b, c\}| = 2$. So by Corollary 17.1, at most $2 + 6 - 4 = 4$ of the sets can be chosen to have an SDR. But then (A_1, A_2, A_5, A_6) has (a, b, c, d) as an SDR, so 4 is the best you can do.

18. November 1: Networks (Section 13.2)

Definition 18.1. A *network* $N = (V, A, s, t, c)$ is a directed graph (V, A), where V is a set of vertices, and A is a collection of ordered pairs ("arcs") of V, with two distinct vertices s (source) and t (target), in which each arc α has a non-negative weight $c(\alpha)$ (the *capacity* of α).

Example. Let $V = \{s, x, y, z, w, t\}$ and $A = \{(s, x), (s, y), (x, y), (x, z), (z, w), (y, w), (z, t), (w, t)\}$. Then the following graph is an example of a directed graph.
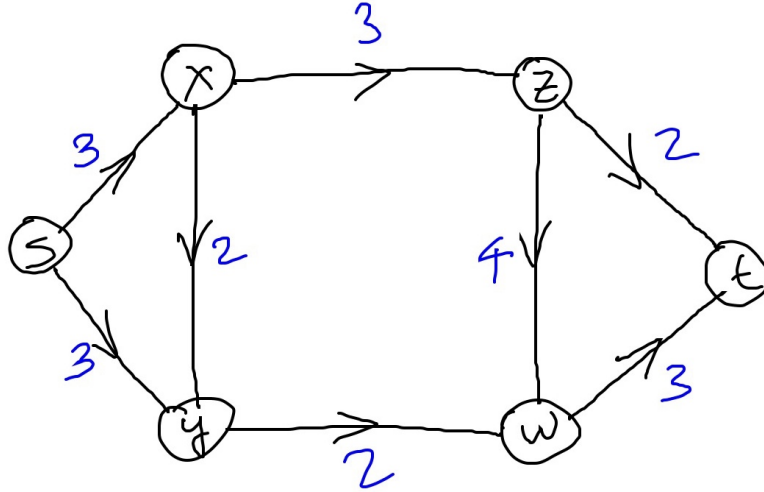
Definition 18.2. A *flow* f on network N is a function $f : A \to \mathbb{R}_{\geq 0}$ such that:

    (i) $0 \leq f(\alpha) \leq c(\alpha)$ for all $\alpha \in A$

    (ii) Let $i(\alpha)$ be an initial vertex of α and $\tau(\alpha)$ a terminal vertex of α. Then $\displaystyle\sum_{i(\alpha)=x} f(\alpha) =$

$\displaystyle\sum_{\tau(\alpha)=x} f(\alpha)$ for all $x \neq s, t$, i.e., flow in $=$ flow out.

Let U be a subset of vertices containing s but not t, and define

$$\overrightarrow{U} = \{\alpha : i(\alpha) \in U, \tau(\alpha) \notin U\}$$
$$\overleftarrow{U} = \{\alpha : \tau(\alpha) \in U, i(\alpha) \notin U\}.$$

Example. Using the same graph in the first example, we see that $\overrightarrow{U} = \{(s,y), (x,z), (w,t), (x,y)\}$ and $\overleftarrow{U} = \{(y,w), (z,w)\}$.

Lemma 18.1. *Let f be a flow in network $N = (V, A, s, t, c)$ and let U be a set of vertices containing s but not t. Then*

$$\sum_{\alpha \in \overrightarrow{U}} f(\alpha) - \sum_{\alpha \in \overleftarrow{U}} f(\alpha) = \underbrace{\sum_{i(\alpha)=s} f(\alpha) - \sum_{\tau(\alpha)=s} f(\alpha)}_{value\ of\ the\ flow}.$$

Proof. Consider

$$\sum_{x \in U} \left(\sum_{i(\alpha)=x} f(\alpha) - \sum_{\tau(\alpha)=x} f(\alpha) \right). \tag{$\star$}$$
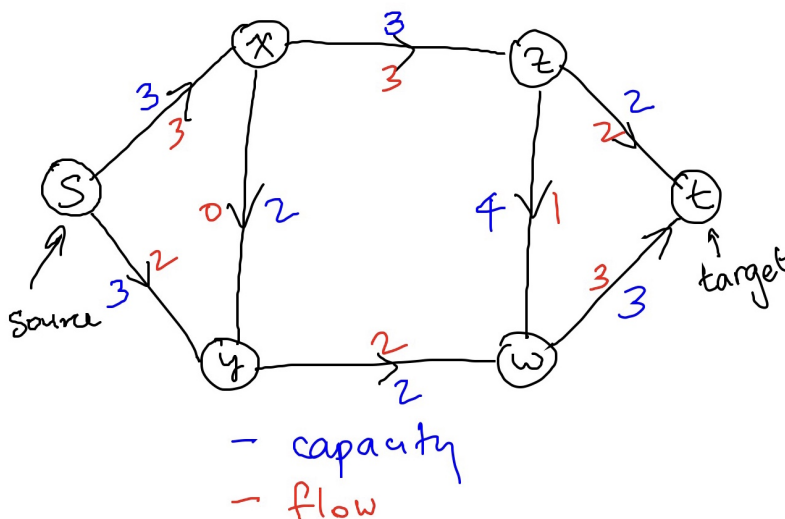
On one hand,

$$(\star) = \sum_{i(\alpha)=s} f(\alpha) - \sum_{\tau(\alpha)=s} f(\alpha).$$

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On the other hand,

$$(\star) = \sum_{x\in U}\sum_{i(\alpha)=x} f(\alpha) - \sum_{x\in U}\sum_{\tau(\alpha)=x} f(\alpha)$$

$$= \sum_{i(\alpha)\in U} f(\alpha) - \sum_{\tau(\alpha)\in U} f(\alpha)$$

$$= \sum_{\alpha\in\overrightarrow{U}} f(\alpha) - \sum_{\alpha\in\overleftarrow{U}} f(\alpha). \qquad \square$$

Example. Flow in has to be equal to flow out for any point.

— capacity
— flow

Definition 18.3. A *max flow f* is a flow that has the largest value.

In case where all capacities are integers, we find the maximum flow (it will have all integer flows on all arcs).

Definition 18.4. A *cut* in a network $N = (V, A, s, t, c)$ is a set C of arcs such that path from s to t contains at least one arc in C. The *capacity* $\text{cap}(C)$ *of cut* C is the sum of the capacities in C. A cut is a *minimum cut* provided it has the smallest capacity amongst all cuts in N.

Lemma 18.2. *Every minimal cut is of the form $\overrightarrow{U}$ for some set U of vertices containing s but not in t.*

Proof. This is Lemma 13.2.2 in the text. $\qquad \square$

Theorem 18.1. *Let $N = (V, A, s, t, c)$ be a network. Then the maximum value of a flow in N is the maximum capacity of a cut in N (i.e., capacity of some minimum cut). If all capacities on all arcs are integers, then there is a maximum flow in which all values are integers.*

Proof (sketch). For each flow and each cut C, we show that $\text{val}(C) \leq \text{cap}(C)$. Without loss of generality, C is of the form $\overrightarrow{U}$.

$$\text{val}(f) = \sum_{i(\alpha)=s} f(\alpha) - \sum_{\tau(\alpha)=s} f(\alpha)$$

$$= \sum_{\alpha \in \overrightarrow{U}} f(\alpha) - \sum_{\alpha \in \overleftarrow{U}} f(\alpha)$$

$$\leq \sum_{\alpha \in \overrightarrow{U}} f(\alpha) \leq \sum_{\alpha \in \overrightarrow{U}} c(\alpha) = c(\overrightarrow{U}).$$

Next show, we show that there exist a flow $\hat{f}$ and a cut $\hat{C}$ such that $\text{val}(\hat{f}) = \text{cap}(\hat{C})$. It then follows that $\hat{f}$ is a maximum flow, and $\hat{C}$ is a minimum cut.. We will prove this by displaying an algorithm that gives us this output.

(1) Start with a flow (the zero flow: 0 on each arc)
(2) We either improve flow, or we stop as we have a max flow:
   (0) $U \leftarrow \{s\}$
   (1) If there exists an arc $\alpha = (x, y)$ with either:
       (i) $x \in U, y \notin U, f(\alpha) < c(\alpha)$; or
   (ii) $x \notin U, y \in U, f(\alpha) > 0$.
   For the first case, put $y \in U$; for the second case, put $x \in U$.
   In the end, output U; if $t \in U$ then we can increase the flow. If $t \notin U$, then we achieve a maximum flow. $\qquad\square$

One interesting application of flow is finding an SDR. Consider $S_1 \cup \cdots \cup S_n = \{x_1, \ldots, x_m\}$. Construct a digraph from $S_1, \ldots, S_n$ to $x_1, \ldots, x_m$, where S_k and x_i are connected (from S_k to x_i) if and only if $x_i \in S_k$. Let s be the source (and flow out to each of S_k); and let t be the target (flow into t from x_i). Let each capacity be 1. Then there exists a maximum flow with all integer flows on all arcs. In fact, any good flow represents a SDR. Note that if that is not the case, there exists a point x_i such that more than 1 flows into x_i then to t, so this is not a good flow.

19. November 20

Consider the bipartite graph below, where $y_1 = \{x_2\}, y_2 = \{x_2, x_3\}, y_3 = \{x_1, x_3, x_4\}, y_4 = \{x_2, x_4, x_5, x_6\}, y_5 = \{x_2\}, y_6 = \{x_2\}$. We want to find an SDR or a maximum matching in the given graph. So this is the same problem but phrased differently.
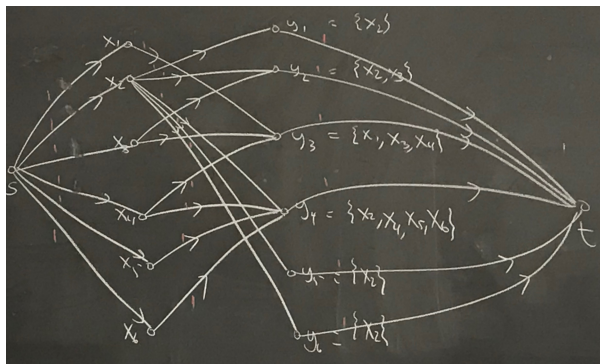
We will see that we can have an SDR for up to four sets. An example would be $(y_1, y_2, y_3, y_4) = (x_2, x_3, x_1, x_4)$.

Definition 19.1. Let $D = (V, A)$ be a digraph (directed graph). Then the *indegree of x* (resp. *outdegree*) in D is

$$\text{indegree}_D \, x = |\{\alpha \in A : \tau(\alpha) = x\}|$$

$$\text{outdegree}_D \, x = |\{\alpha \in A : i(\alpha) = x\}|.$$

Remark. Loops contribute 1 to both the indegree and the outdegree.

Theorem 19.1. *In digraph $D = (V, A)$,*

$$\sum_{x \in V} \text{indegree}_D \, x = \sum_{x \in V} \text{outdegree}_D \, x.$$

Definition 19.2. D is *strongly connected* iff for all $x, y \in V$ there exists a directed path from x to y and a directed path from y to x. A *strongly connected component of D* is a maximal strongly connected subdigraph. The *condensation of digraph D $\rho(D)$* has as its vertices the strongly connected components $c_1, c_2, \ldots, c_k$ of D with an arc from c_i to c_j if and only if there is an arc in D, say $\alpha = (x, y)$ where $x \in c_i$ and $y \in c_j$.

Definition 19.3. A *tournament* is an orientation of K_n.

Theorem 19.2. *In a tournament $T = (V, A)$, there is a vertex v such that for any other vertex x, either v "beats" x or there is a vertex y such that v beats y, and y beats x.*

Proof. Pick a vertex v that beats the most other vertices. We will prove that this v satisfies the desired properties. $\square$

Definition 19.4. A *transitive tournament* is a tournament whose arc set is transitive (if $(x, y), (y, z) \in A$, then $(x, z) \in A$). So if a tournament is transitive, then we can list the vertices as $p_1, \ldots, p_n$ such that p_i beats all to the right of it (note that the tournament is a linear total order).

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