MATH 5020: ANALYTIC FUNCTION THEORY (COMPLEX ANALYSIS)

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1. January 7

Complex analysis studies the functions from $\mathbb{C}$ to $\mathbb{C}$, the set of complex numbers. One can view $\mathbb{C} \cong \mathbb{R}^2$ with the usual vector addition and the usual real-scalar multiplication. Thus we can map each complex number in the usual xy-plane, where the x-axis is the real axis, and the y-axis the complex axis. Clearly, $(x, 0)$ corresponds to the real number x, and $(0, y)$ corresponds to the imaginary number yi. Thus $(x, y) = x(1, 0) + y(0, 1)$ corresponds to $x + iy \in \mathbb{C}$.

Furthermore, we define a product $(x_1 + iy_1)(x_2 + iy_2)$ such that i satisfies $i^2 = -1$, and the usual rules of arithmetic are hold. Thus, we must have $(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_!y_2 + iy_1x_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$. All in all, we have the following definition of $\mathbb{C}$.

Definition 1.1. $\mathbb{C}$, *the set of complex numbers*, is a two-dimensional real vector space ($\mathbb{R}^2$) with the vector addition

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and the vector product

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

With these operations ,$\mathbb{C}$ forms a field.

The real axis is a subfield (or a subspace) identical to $\mathbb{R}$. Note that $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $(x_1, 0) \cdot (x_2, 0) = (x_1x_2, 0)$. Therefore, $\mathbb{R} \subset \mathbb{C}$ by writing point $(x, 0)$ as x (i.e., identify the real axis as $\mathbb{R}$). And the real-scalar multiplication is special case of product $(x_1, 0) \cdot (x_2, y_2) = (x_1x_2, x_1y_2) = x_1(x_2, y_2)$. Hence the following notation is justified: $(x, y) = (x, 0) + (0, y) = x + y(0, 1) = x + iy$.

Remark. $\mathbb{C}$ is a field obtained from $\mathbb{R}$ by adjoining an element i such that $i^2 = -1$. (i.e., $\mathbb{C} = \mathbb{R}(i)$). Note that $x^2 + 1$ is irreducible, and $i^2 + 1 = 0$, so i is degree 2 over $\mathbb{R}$. Hence $\mathbb{C} = \mathbb{R}(i)$ is a vector space over $\mathbb{R}$ of dimension 2, i.e., $\mathbb{C}$ is the smallest field (unique up to isomorphism) containing $\mathbb{R}$ and i where $i^2 = -1$.

Definition 1.2. Let $z = x + iy \in \mathbb{C}$. Then the *real part of* z is $\mathrm{Re}(z) := x$; the *imaginary part of* z is $\mathrm{Im}(z) := y$. Therefore, $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ and $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$ are both coordinate projections $\pi_1 = \mathrm{Re}$ and $\pi_2 = \mathrm{Im}$, which are both real linear maps.

As in any field, there must be an inverse $-z$ and a multiplicative inverse z^{-1}, and they are unique. Define

$$z - w := z + (-w)$$

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$$\frac{z}{2} := zw^{-1}\ (w \neq 0).$$

Then $-z = (-1)z$ and $z^{-1} = 1/z$ for any $z \neq 0$.

What about the integer powers of complex numbers? Let $z^n := \underbrace{z \cdot \cdots \cdot z}_{n \text{ times}}$ and $z^{-n} := \underbrace{z^{-1} \cdot \cdots \cdot z^{-1}}_{n \text{ times}}$. Then for all $m, n \in \mathbb{Z}$, we have

$$\begin{aligned} z^{m+n} &= z^m z^n \\ (z^m)^n &= z^{mn}. \end{aligned}$$

Furthermore, we let $0^0 = 1$ and $0^n = 0$ for all $n \in \mathbb{N}$.

A norm can be defined on $\mathbb{C}$ to make $\mathbb{C}$ a metric space. For any $z = x + iy \in \mathbb{C}$, define

$$|z| := \|(x, y)\|_{\text{Euc}} = \sqrt{x^2 + y^2}.$$

One can easily verify that $|z|$ indeed has the norm properties, i.e.

- $|z| = 0 \Leftrightarrow z = 0$
- $|z + w| \leq |z| + |w|$
- $|az| = |a||z|$ where $a \in \mathbb{R}$.

Additionally, $|z|$ satisfies $|wz| = |w||z|$ and hence

$$\left|\frac{w}{z}\right| = \frac{|w|}{|z|}$$

and

$$|z^{-1}| = \left|\frac{1}{z}\right| = \frac{1}{|z|} = |z|^{-1}.$$

Definition 1.3. Write $\overline{z} := x - iy$ for any $z = x + iy$. Then $\overline{z}$ is said to be the *conjugate of* z. In other words, the map $z \mapsto \overline{z}$ is a reflection in the x-axis.

The following are the properties of the conjugation map:

- Real-linear: $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{az} = a\overline{z}$ for $a \in \mathbb{R}$.
- Involution: bijective, and the inverse is itself.
- Isometry: $|\overline{z}| = |z|$.
- Preserves the product: $\overline{zw} = \overline{zw}$.

All in all, the conjugation map is a field isomorphism. Thus we also have

- $\overline{z - w} = \overline{z} - \overline{w}$
- $\overline{z/w} = \overline{z}/\overline{w}$
- $\overline{z^{-1}} = \overline{z}^{-1}$.

Notice also that

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2 \\ \frac{1}{z} &= \frac{\overline{z}}{|z|^2}\ (z \neq 0). \end{aligned}$$

Furthermore,

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$$

$$\text{Im} z = \frac{1}{2i}(z - \overline{z}).$$

1.1. **Polar coordinates**

Given $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, there exists unique $r > 0$ and angle θ (unique mod 2π) so that $x = r\cos\theta$ and $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2} = |z|$ (the length of the vector (x, y)). As for the angle, θ must satisfy $\tan\theta = y/x$ with the quadrant same as (x, y).

Definition 1.4. θ as defined above is called an *argument of* z, which we write $\arg(z)$. Thus,

$$\arg(z) = \{\text{all such values of } \theta \in \mathbb{R}\}.$$

The *principal value of the argument* z, written $\text{Arg}(z)$, is the unique value of θ in $\arg(z)$ such that $\theta \in (-\pi, \pi]$. In other words,

$$\arg(z) = \{\text{Arg}(z) + 2k\pi : k \in \mathbb{Z}\}.$$

Remark. Notice that $\arg : z \mapsto \arg(z)$ is not a function (because arg is multiple-valued). However, $\text{Arg} : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ is a well-defined function.

Notice for $w = a + ib = r(\cos\theta + i\sin\theta)$ and $z = x + iy$,

$$wz = (a + ib)(x + iy) = (ax - by) + i(bx - ay) \leftrightarrow (ax - by, bx + ay),$$

and so

$$\begin{aligned} \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

for $w = r(\cos\theta + i\sin\theta)$. Thus the multiplication in polar form is

$$r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)). \tag{1}$$

In other words, $|z_1 z_2| = |z_1||z_2|$; per the trigonometric identities, $\arg(z_1 z_2) = \arg z_1 + \arg z_2 \bmod 2\pi$. Putting these facts together, we have

$$wz = M_w \begin{pmatrix} x \\ y \end{pmatrix},$$

where M_w denotes the scaled rotation rR_θ. This yields the correspondence between $\mathbb{C}$ and the matrix of scaled rotations, namely

$$w := a + ib = r(\cos\theta + i\sin\theta) \in \mathbb{C} \mapsto M_w := \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = rR_\theta.$$

In fact, this correspondence is a field isomorphism.

1.2. **The unit circle**

Let $S = \{z : |z| = 1\} = \{x + iy : x^2 + y^2 = 1\} = \{\cos\theta + i\sin\theta : \theta \in \mathbb{R}\}$. This is a subgroup of $(\mathbb{C} \setminus \{0\}, \cdot)$.

Proposition 1.1 (Euler's formula). $e^{i\theta} = \cos\theta + i\sin\theta$.

Using Euler's formula, we see that $S = \{e^{i\theta} : \theta \in \mathbb{R}\}$.

From Euler's formula, we see that $\mathbf{e}^{i\theta} = 1$ if and only if $\cos\theta = 1$ and $\sin\theta = 0$. Therefore $\theta = 2\pi k$ for $k \in \mathbb{Z}$. Therefore $\theta \in 2\pi\mathbb{Z}$.

Clearly, the conjugate of $e^{i\theta}$ is $\overline{e^{i\theta}} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$. As we have seen in (1), we have $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, so $(e^{i\theta})^{-1} = e^{-i\theta}$. Therefore by induction, for $n \in \mathbb{Z}$, $(e^{i\theta})^n = e^{in\theta}$.

So the map $\varphi : \mathbb{R} \to S$ given by $\theta(t) = e^{it}$ maps the real line to the unit circle. Note that φ is a group homomorphism, since $\varphi(s+t) = \varphi(s)\varphi(t)$ for all $s, t \in \mathbb{R}$. Furthermore, notice that $\ker\varphi = \varphi^{-1}(1) = 2\pi\mathbb{Z} = \arg(1)$, and the cosets are

$$\arg(e^{i\theta}) = \varphi^{-1}(e^{i\theta}) = \theta + 2\pi\mathbb{Z}.$$

1.3. **Complex numbers in polar form**

For any $z = x + iy \in \mathbb{C} \setminus \{0\}$, z has the polar form

$$z = r\cos\theta + ri\sin\theta = re^{i\theta},$$

is $r > 0$. So (r, θ) is the polar coordinate for (x, y). Indeed, in this case we let $r = |z|$ and $\theta \in \arg z$, which is unique up to mod 2π. That is, in polar forms, $r_1e^{i\theta_1} = r_2e^{i\theta_2}$ if and only if $r_1 = r_2$ and $\theta_1 \equiv \theta_2 \pmod{2\pi}$.

As for the product, note that $(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = (r_1r_2)e^{i(\theta_1+\theta_2)}$. If $z = re^{i\theta}$, then $\overline{z} = re^{-i\theta}$ and $z^{-1} = r^{-1}e^{-i\theta}$. This gives us de Moivre's law.

Proposition 1.2 (De Moivre's law)**.** *For any $n \in \mathbb{Z}$, we have $z^n = r^ne^{in\theta}$.*

We can use de Moivre's law to find the n-th roots of $z = re^{i\theta} \in \mathbb{C} \setminus \{0\}$. For $n \in \mathbb{N}$, there exists exactly n numbers $w \in \mathbb{C}$ such that $w^n = z = re^{i\theta}$. By de Moivre's law, they are

$$w = r^{1/n}e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)},$$

for $k = 0, 1, 2, \ldots, n-1$.

Definition 1.5. The *ray of θ* is $\mathrm{Ray}_\theta = \{re^{i\theta} : r \geq 0\}$.

1.4. **Complex exponentials and the exponential map**

For $z = x + iy \in \mathbb{C}$ we define

$$e^z = e^{x+iy} = e^xe^{iy} = e^x\cos y + ie^x\sin y.$$

Thus $\mathrm{Re}(e^z) = e^x\cos y$ and $\mathrm{Im}(e^z) = e^x\sin y$. Notice that $|e^z| = e^x > 0$, so e^z can never be 0. Finally, $\arg(e^z) = y \mod 2\pi$. As for the conjugate and the inverse,

$$\overline{e^z} = \overline{e^xe^{iy}} = e^xe^{-iy} = e^{x-iy} = e^{\overline{z}}$$
$$(e^z)^{-1} = e^{-z}.$$

Proposition 1.3. *For any $z, w \in \mathbb{C}$ and $n \in \mathbb{Z}$,*

$$e^ze^w = e^{z+w}$$
$$(e^z)^n = e^{nz}.$$

When is $e^z = 1$? Recall that $e^z = e^xe^{iy} = 1$, so $|e^z| = e^x = 1$ and $e^{iy} = 1$. This means $x = 0$, and $y = 2\pi k$ where $k \in \mathbb{Z}$. Hence $z = 2\pi ik$ for $k \in \mathbb{Z}$, so $z \in 2\pi i\mathbb{Z}$.

When do we have $e^z = e^w$? if $e^z = e^w$, then $e^ze^{-w} = e^{z-w} = 1$. Thus $z - w \in 2\pi i\mathbb{Z}$.

Definition 1.6. The *complex exponential mapping* $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is defined as $\exp(z) = e^z$ (or equivalently, $\exp(x + iy) = e^x e^{iy}$).

Now let's look at the vertical lines in the complex plane (i.e., real part fixed, imaginary part varies). If the real part is 0, then exp maps the points to the unit circle. Thus the imaginary axis is mapped to the unit circle. The set of complex numbers with $\operatorname{Re}(z) = a$, is mapped to the circle of radius e^a.

Take a look at the real axis (i.e., $\operatorname{Im}(z) = 0$). Then exp maps the real axis to the positive real axis, with smaller z being mapped closer to 0. Any complex number such that $\operatorname{Im}(z) = a$, is mapped to a point lying on the half-line whose angle between the positive real axis is exactly a.

Another topic of interest for any function is the pre-image of the function. Let $w = re^{i\theta} \in \mathbb{C} \setminus \{0\}$. Then we want to find $z = x + iy$ such that $e^z = w = re^{i\theta}$. Recall that the modulus is dictated by the value of x, so $r = e^x$. Hence $x = \ln r$. On the other hand, $e^{i\theta}$ advises the angle, so we need $e^{i\theta} = e^{iy}$. This means $y = \theta + 2\pi k$ for $k \in \mathbb{Z}$, i.e., $y \in \arg w$. Hence

$$\exp^{-1}(w) = \{\ln|w| + i\alpha : \alpha \in \arg w\} = \ln|w| + i \arg w.$$

1.5. **Complex logarithm**

For $w \in \mathbb{C} \setminus \{0\}$, we define

$$\log w := \exp^{-1}(w) = \ln|w| + i \arg w.$$

Notice this is a multiple-valued function. We can convert this one-to-one by taking advantage of the principal argument $\operatorname{Arg} w$.

Definition 1.7. The *principal value of the logarithm*, denoted $\operatorname{Log} w$, is

$$\operatorname{Log} w := \ln|w| + i \operatorname{Arg} w.$$

Therefore, $\log w = \ln|w| + i \operatorname{Arg} w + 2\pi i\mathbb{Z} = \operatorname{Log} w + 2\pi i\mathbb{Z}$.

2. January 9: Metric spaces

Let (X, d) be a metric space, with a metric $d(x, y)$.

Definition 2.1. The *δ-ball centred at a* is

$$B(a, \delta) := \{x \in X : d(a, x) < \delta\}.$$

Example. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Then δ-balls are open intervals. $\mathbb{C} \cong \mathbb{R}^2$ with the usual Euclidean metric is another example of a metric space.

Definition 2.2. The *annulus* $A(a, r, R)$ for any $r < R$ is

$$A(a, r, R) := \{x \in X : r < d(a, x) < R\}.$$

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For $f : A \to \mathbb{C}$ where A is $\mathbb{R}$ or $\mathbb{C}$, write

$$f(z) = u(z) + iv(z) = \begin{pmatrix} u(z) \\ v(z) \end{pmatrix}.$$

Then

$$\lim_{z\to a} f(z) = \begin{pmatrix} \lim\limits_{z\to a} u(z) \\ \lim\limits_{z\to a} v(z) \end{pmatrix} = \lim_{z\to a} u(z) + i \lim_{z\to a} v(z).$$

Thus $\lim f(z)$ exists if and only if both $\lim u(z)$ and $\lim v(z)$ exist.

Definition 3.1. The function $f : \mathbb{R} \to \mathbb{R}$ is *differentiable at a* if

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists.

What if the codomain $\mathbb{R}$ is replaced with $\mathbb{R}^2$? Consider $\gamma : \mathbb{R} \to \mathbb{R}^2$.

Definition 3.2. Let $\gamma : \mathbb{R} \to \mathbb{R}^2 = \mathbb{C}$ be equal to

$$\gamma(t) = u(t) + iv(t).$$

Then γ is *differentiable at a* if

$$\gamma'(a) = \lim_{t\to a} \frac{\gamma(t) - \gamma(a)}{t - a}$$

exists.

Remark. $\gamma'(a)$ from the above definition exists if and only if both $u'(a)$ and $v'(a)$ exist, since

$$\gamma'(a) = \begin{pmatrix} \lim\limits_{t\to a} \frac{u(t)-u(a)}{t-a} \\ \lim\limits_{t\to a} \frac{v(t)-v(a)}{t-a} \end{pmatrix} = \begin{pmatrix} u'(a) \\ v'(a) \end{pmatrix}$$

$$= \lim_{t\to a} \frac{u(t) - u(a)}{t - a} + i \lim_{t\to a} \frac{v(t) - v(a)}{t - a} = u'(a) + iv'(a).$$

Therefore γ is differentiable at a if and only if u and v are differentiable at a.

Notice that one equivalent way of defining differentiability involves using the linear map. It may sound unnecessarily convoluted, but understanding this perspective is imperative since this perspective allows us to generalize differentiability for $\mathbb{R}^n$ for all $n \geq 1$.

Definition 3.3. $f : \mathbb{R} \to \mathbb{R}$ is *differentiable at a* if there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{t\to a} \frac{|f(t) - f(a) - \alpha(t - a)}{|t - a|} = 0.$$

Notice that if we define $L : \mathbb{R} \to \mathbb{R}$ where $L(t) := \alpha t$, then we can also write that

$$\lim_{t\to a} \frac{|f(t) - f(a) - L(t - a)|}{|t - a|} = 0.$$

Therefore, we can generalize this for any multi-dimensional functional $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n, m \in \mathbb{N}$. From now on, $\|\cdot\|$ shall denote the Euclidean norm.

Definition 3.4. $f : \mathbb{R}^n \to \mathbb{R}^m$ is *real-differentiable at a* if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{v}\to\mathbf{a}} \frac{\|f(\mathbf{v}) - f(\mathbf{a}) - L(\mathbf{v} - \mathbf{a})\|}{\|\mathbf{v} - \mathbf{a}\|} = 0.$$

If such L exists, then L is unique; this unique L is the *full derivative of f at a*, which we denote $Df(a)$.

Since $\mathbb{C} \cong \mathbb{R}^2$, mostly we will focus on the $n = m = 2$ case. For $f : \mathbb{R}^2 \to \mathbb{R}^2$, the linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ has a matrix representation, namely

$$[Df(a)] = \begin{pmatrix} \frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a) \\ \frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a) \end{pmatrix},$$

where $f = u + iv = \begin{pmatrix} u \\ v \end{pmatrix}$.

For $f : \mathbb{R}^2 \to \mathbb{R}$, the linear map $Df(a) : \mathbb{R}^2 \to \mathbb{R}$ has a 1×2 matrix

$$[Df(a)] = \begin{pmatrix} \frac{\partial f}{\partial x}(a) & \frac{\partial f}{\partial y}(a) \end{pmatrix}.$$

In this case, we define the gradient function

$$\nabla f(a) = \begin{pmatrix} \frac{\partial u}{\partial y}(a) & \frac{\partial u}{\partial y}(a) \end{pmatrix}.$$

Example. Any constant functions are differentiable. If $f \equiv C$, then $Df(x) = \mathbf{0}$ everywhere where f is defined.

Example. Linear maps are always differentiable. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $DT(x) \equiv T$ for all $x \in \mathbb{R}^n$.

3.1. **Test for differentiability**

Definition 3.5. If the partial derivatives of f exists and are continuous on an open set U, then we say that f *is differentiable at* U *(or* C^1 *on* U*)*.

Proposition 3.1. *Let* f *be differentiable at an open set* $U \subseteq \mathbb{R}$ *(or* $\mathbb{R}^2$*).*

(1) Differentiability implies continuity.
(2) (Chain rule) If f *(resp.* g*) is differentiable at* a *(resp.* $f(a)$*), then* $g \circ f$ *is differentiable at* a *also; and*

$$D(g \circ f)(a) = Dg(f(a))Df(a).$$

Example. If $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are both differentiable, then $(g \circ f)'(a) = \nabla g(f(a)) \cdot f'(a) \cdot f'(a)$, where $\cdot$ denotes the dot product.

Theorem 3.1 (Inverse function theorem)**.** *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *where* f *is defined on an open set* U*, and let* $a \in U$*. If* f *is* C^1 *in* U*, and* $Df(a)$ *is invertible, then there exist open sets* U_0 *and* V_0 *containing* a *and* $f(a)$*, respectively, such that:*

(1) $f : U_0 \to V_0$ *is invertible whose inverse* $f^{-1} : V_0 \to U_0$ *is* C^1 *as well, and*
(2) $Df^{-1}(v) = (Df(u))^{-1}$ *for* $v = f(u) \in V$ *and* $u \in U$*.*

In other words, f *is locally invertible with an inverse that is also* C^1*.*

4. January 14

4.1. **Paths**

Definition 4.1. Let $E \subseteq \mathbb{C}$ and $p, q \in E$. A *path* from p to q is a continuous map $\gamma : [a, b] \to \mathbb{C}$ defined by $t \mapsto \gamma(t) = x(t) + iy(t)$ such that $\gamma(a) = p, \gamma(b) = q$, and $\gamma([a, b]) \subset E$. Therefore both $x(t)$ and $y(t)$ must be continuous (because that is equivalent to $\gamma(t)$ being continuous). γ is a *differentiable path* if $\gamma : [a, b] \to \mathbb{C} \cong \mathbb{R}^2$ is differentiable. $\gamma : [a, b] \to \mathbb{C}$ is a C^1 path if γ is continuous and γ' is continuous. γ is a *smooth path* if γ is a C^1 path and

$\gamma' \neq 0$ on $[a, b]$ (so as to remove any singularity). γ is a *piecewise-C^1 path* if γ is C^1 on each subinterval of some partition $a = t_0 < t_1 < \cdots < t_n = b$. Similarly, γ is a *piecewise smooth path* if γ is smooth on each subinterval of some partition $a = t_0 < t_1 < \cdots < t_n = b$.

One particular type of paths of our interest is a *line-segment path*, i.e., $\gamma : [0, 1] \to E$ where $\gamma(t) = z + t(w - z)$ so that $\gamma(0) = z$ and $\gamma(1) = w$. Putting multiple line-segment paths together, we have *polygonal paths* or *rectangular paths*.

Definition 4.2. A set E is *connected* if $E \neq A \cup B$ for non-empty separated sets A and B (i.e., $A \cap B = \emptyset, A \subset B_{\text{ext}}$ and $B \subset A_{\text{ext}}$).

Example. Disjoint open sets are separated sets. The empty set is separated from every set.

Theorem 4.1. *Let E be a connected set, and f be a continuous map on E. Then $f(E)$ is also connected.*

Definition 4.3. A set E is *path-connected* if for any $p, q \in E$ there exists a path in E from p to q. Similarly, we also have analogous definitions for *smooth-path-connectedness, piecewise-smooth-connectedness, rectangular-path-connectedness*, and so forth.

Theorem 4.2. *Path-connectedness implies connectedness.*

Proof (sketch). Suppose $E = A \cup B$ where A and B are separated, and that $p \in A$ and $q \in B$. But since E is path-connected, there exists a path from γ in E from p to q. Thus $\gamma : [a, b] \to E$ is continuous. But then $[a, b]$ is continuous, so $\gamma([a, b])$ must be connected. But this contradicts the fact that E can be written in two separated sets. □

Remark. Note that rectangular-path-connected $\Rightarrow$ polygonal-path-connected $\Rightarrow$ piecewise-smooth-connected $\Rightarrow$ path-connected $\Rightarrow$ connected. Also, smooth-path-connectedness $\Rightarrow$ piecewise-smooth-connectedness.

Definition 4.4. A *domain* (or *region*) is a connected open set in $\mathbb{C}$.

Theorem 4.3. *Every domain $\mathcal{U}$ is path-connected (for each above type of path).*

Proof (sketch). Let $\mathcal{U}$ be a connected open set, and let $p \in \mathcal{U}$. Let $A = \{u \in U :$ there exists a rectangular path in $\mathcal{U}$ from p to $u\}$. Then A is open. However, $B = \mathcal{U} \setminus A$ is also open since we can apply the same reasoning used to argue that A is open. Hence A and B are separated, but this contradicts the fact that $\mathcal{U}$ is connected. Hence $A = \mathcal{U}$. □

Theorem 4.4. *If $\mathcal{U}$ is a domain and rectangular-path-connected, then it is smooth-path-connected.*

Example. Disks $D(z_0, r)$ are domains. Annuli $A(z_0, r, R)$ are domains. The upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ is a domain. Slice planes $\mathbb{C} \setminus \text{Ray}_\theta$ are domains as well.

Theorem 4.5 (Zero derivative theorem)**.** *If $f' \equiv 0$, continuous on $[a, b]$ and differentiable on (a, b), then f is constant (on $[a, b]$).*

Proof. Apply the mean value theorem. □

We can generalize this to $\mathbb{R}^n$.

Theorem 4.6 (Generalized ero derivative theorem)**.** *Let $f : [a, b] \to \mathbb{R}^n$ be continuous on $[a, b]$, differentiable on (a, b), and $f' \equiv 0$ on (a, b), then f is a constant on $[a, b]$.*

Theorem 4.7. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ differentiable on a domain $\mathcal{U}$ and $Df \equiv 0$ on U, then f is constant on $\mathcal{U}$.*

Proof. Fix $z_0 \in \mathcal{U}$. Then given any $z \in \mathcal{U}$, there exists a smooth path $\gamma : [a,b] \to \mathcal{U}$ in $\mathcal{U}$ from z_0 to z so that $\gamma(a) = z_0$ and $\gamma(b) = z$. So $f \circ \gamma : [a,b] \to \mathbb{R}^2$ is continuous on $[a,b]$, differentiable on (a,b) with $(f \circ \gamma)'(t) = Df(\gamma(t))\gamma'(t) = 0$ since $Df(\gamma(t)) = 0$. Hence $f \circ \gamma$ is constant on $[a,b]$. Thus $f(z_0) = (f \circ \gamma)(a) = (f \circ \gamma)(b) = f(z)$. Since z is arbitrary, indeed $f \equiv f(z_0)$. □

4.2. **Holomorphic (complex-differentiable) functions**

Let $f : \mathbb{C} \to \mathbb{C}$ be defined on an open set $\mathcal{U}$, and write $f(z) = u(z) + iv(z)$. If $z = x + iy$, then

$$f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$$

Definition 4.5. f is *holomorphic* (or *complex-differentiable*) at $a \in \mathcal{U}$ if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists. If so, we denote its value by $f'(a)$ or $\frac{df}{dz}(a)$. Thus, $f'(a)$ is a complex number.

Also, as in $\mathbb{R}$,

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \alpha \in \mathbb{C} \Leftrightarrow \lim_{z \to a} \frac{|f(z) - f(a) - \alpha(z-a)|}{|z-a|} = 0.$$

So there is a matrix M_α so that $\alpha(z-a) = \alpha(z-a)$.

$$M_\alpha(z) = \alpha z = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = rR_\theta \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\alpha = c + id, r = |\alpha|, \theta = \operatorname{Arg} \alpha$. Hence $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at a. Thus, we see that $f : \mathbb{C} to \mathbb{C}$ is holomorphic at a if and only if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is real-differentiable at a, and $Df(a)$ is a scaled rotation.

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Let $f : \mathbb{C} to \mathbb{C}$ be $f(z) = u(z) + iv(z)$, where $z = x + iy$. Then using the partials, we see that

$$Df(a) = \frac{\partial u}{\partial x}(a) = \begin{pmatrix} \frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a) \\ \frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a) \end{pmatrix} = M_{f'(a)},$$

so the matrix must be of the form $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$. This gives us the Cauchy-Riemann equation.

Theorem 5.1 (Cauchy-Riemann equations)**.** *$f : \mathbb{C} \to \mathbb{C}$ is holomorphic at a if and only if*

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \text{ and } \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a).$$

So if f is holomorphic at a, then

$$f'(a) = \frac{\partial u}{\partial x}(a) + i\frac{\partial v}{\partial x}(a) = \frac{\partial u}{\partial x}(a) - i\frac{\partial u}{\partial y}(a) = \frac{\partial v}{\partial y}(a) + i\frac{\partial v}{\partial x}(a) = \frac{\partial v}{\partial y}(a) - i\frac{\partial u}{\partial y}(a).$$

Locally speaking, a holomorphic map is like a scaled rotation for any z near a, i.e.,

$$f(z) - f(a) \approx f'(a)(z - a).$$

Definition 5.1. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at each $z \in \mathcal{U}$ for an open set $\mathcal{U}$, then f is *holomorphic on* $\mathcal{U}$. If f is holomorphic on $\mathbb{C}$, then f is an *entire function*.

Theorem 5.2. *If all four partials of $f = u + iv$ exist, are continuous, and satisfy the Cauchy-Riemann equations on $\mathcal{U}$, then f is holomorphic on $\mathcal{U}$, and f' is continuous.*

5.1. **Properties of holomorphic functions**

Proposition 5.1. *If f is holomorphic at a, then f is continuous at a.*

Theorem 5.3. *If f and g are holomorphic on $\mathcal{U}$ then the sum $f + g$ is holomorphic on $\mathcal{U}$, and*

$$(f + g)'(z) = f'(z) + g'(z).$$

Similarly, fg is holomorphic on $\mathcal{U}$ and

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

f/g is holomorphic on $\mathcal{U} \setminus \{a \in \mathcal{U} : g(a) = 0\}$, and

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

Proof. The proofs work exactly the same way for the real versions. We will prove the product rule as an example.

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} &= \lim_{z \to z_0} \frac{f(z)g(z) - f(z)g(z_0) + f(z)g(z_0)f(z_0)g(z_0)}{z - z_0} \\
&= \lim_{z \to z_0} f(z)\frac{g(z) - g(z_0)}{z - z_0} + \frac{f(z) - f(z_0)}{z - z_0}g(z_0) \\
&= \lim_{z \to z_0} f(z)g'(z_0) + f'(z_0)g(z).
\end{aligned}$$

But since f and g are holomorphic, they are continuous as well, so $f(z), g(z) \to f(z_0), g(z_0)$ respectively as $z \to z_0$. The result now follows. □

Example. If $f : \mathbb{C} \to \mathbb{C}$ is a constant function for some $a \in \mathbb{C}$ (i.e., $f(z) \equiv a$), then $Df(z) \equiv 0$ which is a scaled rotation. Thus f is holomorphic with $f'(z) \equiv 0$. Another way of looking at it is

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{a - a}{z - z_0} = 0.$$

Thus f is entire, and it has $f' \equiv 0$.

Example. Let $f(z) = z$. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1.$$

Therefore f is entire with $f' \equiv 1$.

Example. Let $n \in \mathbb{Z}$, and $f(z) = z^n$. Then $f'(z) = nz^{n-1}$ which one can show with the product rule and induction. Similarly, if $f(z) = 1/z$, then f is holomorphic on the punctured complex plane ($\mathbb{C} \setminus \{0\}$), and $f'(z) = -1/z^2$ (use the quotient rule).

With the application of chain rule, one can also show that $(z^{-n})' = -nz^{-n-1}$ for any $z \in \mathbb{C} \setminus \{0\}$.

Example. Any polynomial function $p(z) = a_n z^n + \cdots a_1 z + a_0$ (for $n \in \mathbb{N}, a_n \neq 0$) is entire, and $p'(z) = na_n z^{n-1} + \cdots + a_1$. Any rational function $r(z) = p(z)/q(z)$ is holomorphic in $\mathbb{C} \setminus \{a \in \mathbb{C} : q(x) = 0\}$.

Example. Recall that $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ maps $x + iy \mapsto e^x e^{iy} = e^z$, and that this is differentiable since

$$D\exp(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^x R_y.$$

Thus exp is entire. Finally,

$$\exp'(z) = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^x e^{iy} = e^z = \exp(z).$$

Theorem 5.4 (Chain rule)**.** *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on $\mathcal{U}$ an open set, and $g : \mathbb{C} \to \mathbb{C}$ holomorphic on $\mathcal{V}$ an open set. Suppose also that $f(\mathcal{U}) \subset \mathcal{V}$. Then $g \circ f$ is holomorphic on $\mathcal{U}$ with $(g \circ f)'(z) = g'(f(z))f'(z)$.*

Proof. This follows from the chain rule for functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ which tells us that $g \circ f$ is a real-differentiable function on $\mathcal{U}$, with

$$D(g \circ f)(z) = Dg(f(z))Df(z).$$

Since both $Dg(f(z))$ and $Df(z)$ are scaled rotations, so is $Dg(f(z))Df(z) = D(g \circ f)(z)$. □

Suppose that $\gamma : [a, b] \to C$ is differentiable and $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on an open set $\mathcal{U}$, and that $\gamma([a, b]) \subset \mathcal{U}$. Then using the chain rule, $(f \circ \gamma)'(t) = Df(\gamma(t))\gamma'(t) = f'(\gamma(t))\gamma'(t)$.

Remark. Chain rule holds also for one-sided limits. That is, $(f \circ \gamma)'(a^+) = Df(\gamma(a))\gamma(a^+)$. The same holds for b^-.

Remark. If γ is C^1 or piecewise-C^1, then the same holds for $f \circ \gamma$. If γ is smooth or piecewise-smooth with $f' \neq 0$, then the same claim holds for $f \circ \gamma$.

6. January 18

6.1. Angle-preserving maps

Definition 6.1. A *smooth path at z_0* is a path $\gamma : [-a, a] \to \mathbb{C}$ for some $a > 0$ such that $\gamma(0) = z_0$, γ differentiable at z_0 with $\gamma'(0) \neq 0$. The *angle* between two smooth pats γ and μ at z_0 is the angle between the vectors that are tangent to γ and μ respectively at z_0, i.e., $\measuredangle(\gamma'(0), \mu'(0))$.

Definition 6.2. A map $f : \mathbb{C} \to \mathbb{C}$ defined on an open set $\mathcal{U}$ is *conformal* (or *angle-preserving*) at $z_0 \in \mathcal{U}$ if there exists $\alpha \neq 0$ such that if γ is a smooth path at z_0 in $\mathcal{U}$ then $f \circ \gamma$ is differentiable at 0 and $(f \circ \gamma)'(0) = \alpha\gamma'(0)$ (i.e., scalar rotation).

Theorem 6.1. *f is holomorphic at z_0 with $f'(z_0) \neq 0$ if and only if f is conformal at z_0.*

Proof. ($\Rightarrow$) This is straightforward from the chain rule: $(f \circ \gamma)'(0) = f'(\gamma(0))\gamma'(0) = f'(z_0)\gamma'(0)$, so the result follows upon recognizing that $\alpha = f'(z_0)$.
($\Leftarrow$) Left as an exercise. □

Theorem 6.2 (Inverse function theorem)**.** *Suppose $\mathcal{U}$ is open and $z_0 \in \mathcal{U}$. If f is holomorphic on $\mathcal{U}$, $f'(z_0) \neq 0$, and f' is continuous on $\mathcal{U}$, then there exist open sets $\mathcal{U}_0$ and $\mathcal{V}_0$ with $z_0 \in \mathcal{U}_0 \subset \mathcal{U}$ and $f(z_0) \in \mathcal{V}_0$ such that $f : \mathcal{U}_0 \to \mathcal{V}_0$ is invertible with a holomorphic inverse $f^{-1} : \mathcal{V}_0 \to \mathcal{U}_0$. Furthermore,*

$$f^{-1}(w) = \frac{1}{f('(z)}$$

for all $w = f(z) \in \mathcal{V}_0$ and $z \in \mathcal{U}_0$.

Proof. Since $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 on $\mathcal{U}$ and $Df(z_0) = M_{f'(z_0)}$ is a scaled rotation and invertible (since $f'(z_0) \neq 0$), then the inverse function theorem on $\mathbb{R}^2$ implies that there exist $\mathcal{U}_0$ and $\mathcal{V}_0$ with $z_0 \in \mathcal{U}_0 \subset \mathcal{U}$ and $f(z_0) \in \mathcal{V}_0$ so that $f : \mathcal{U}_0 \to \mathcal{V}_0$ is invertible and $f^{-1} : \mathcal{V}_0 \to \mathcal{U}_0$ is real differentiable. Furthermore, there exists $f(z) := w \in \mathcal{V}_0$ and $z \in \mathcal{U}_0$ so that

$$Df^{-1}(w) = (Df(z))^{-1} = (M_{f'(z)})^{-1} = M_{1/f'(z_0)},$$

which is a scaled rotation. Thus f^{-1} is holomorphic on $\mathcal{V}_0$ with $(f^{-1})'(w) = 1/f'(z)$, as required. □

Theorem 6.3 (Zero derivative theorem)**.** *If f is holomorphic on domains $\mathcal{U}$ and $f' \equiv 0$ on $\mathcal{U}$, then f is constant on $\mathcal{U}$.*

Proof. This is immediate from the analogous result for $\mathbb{R}^2$. □

Definition 6.3. Let $f : \mathbb{C} \to \mathbb{C}$ be a function defined on an open set $\mathcal{U}$. A *primitive* for f on $\mathcal{U}$ is a holomorphic function g on $\mathcal{U}$ such that $g' = f$ on $\mathcal{U}$.

Remark. If g_1 and g_2 are primitives for f on a domain $\mathcal{U}$, then $(g_1 - g_2)' = g_1' - g_2' = f - f = 0$ on $\mathcal{U}$. So by the zero-derivative theorem, $g_1 - g_2 \equiv C$ for some constant C on $\mathcal{U}$. Therefore, $g_1 = g_2 + C$ on $\mathcal{U}$. So if g is a primitive for f on $\mathcal{U}$ open, then so is $g + C$ for any $C \in \mathbb{C}$.

6.2. **Multiple-valued map**

Definition 6.4. Let A, B be non-empty sets. Then a *multiple-valued map* $F : A \to B$ is a function $F : A \to \mathcal{P}(B)$ such that for each a, $F(a) \subset B$. A *representative function* for F on a set E in A is a function $f : A \to B$ such that for each $a \in E$, $f(a) \in F(a)$.

Example. Let $F : [0, \infty) \to \mathbb{R}$ be defined $F(x) = \pm\sqrt{x}$. Then $F(x)$ is a multiple-valued map. One possible representative function is

$$f(x) = \begin{cases} \sqrt{x} & x \in \mathbb{Q} \\ -\sqrt{x} & x \notin \mathbb{Q}. \end{cases}$$

Definition 6.5. A *branch of F on E* is a continuous representative function on a connected open set E.

Example. $\sqrt{x}$ is a branch of F on $(0, \infty)$.

Example. $\arg : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is multiple-valued; one representative function of arg is the principal argument $\mathrm{Arg} : \mathbb{C}\setminus\{0\} \to [-\pi, \pi)$. Note that we need to restrict the domain further to make Arg continuous by replacing $\mathbb{C} \setminus \{0\}$ with $\mathbb{C} \setminus \mathrm{Ray}_\pi$. Therefore $\mathrm{Arg} : \mathbb{C} \setminus \mathrm{Ray}_\pi \to (-\pi, \pi)$ is a branch of arg on $\mathbb{C} \setminus \mathrm{Ray}_\pi$. This is also called the *principal branch*. Similarly, for Ray_θ, the function $\arg : \mathbb{C} \setminus \mathrm{Ray}_\theta \to (\theta, \theta + 2\pi)$ is a branch of arg.

6.3. **Multiple-valued inverses**

Let X and Y be non-empty sets, and suppose that $g : X \to Y$ is a function that is not one-to-one. Then its pre-images

$$g^{-1}(y) = \{x \in X : g(x) = y\}$$

define a multiple-valued map.

Definition 6.6. The function $F : Y \to X$ such that $y \mapsto F(y) =: g^{-1}(y)$ is called the *multiple-valued inverse* of g.

Example. $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a multiple-valued inverse of $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$.

Definition 6.7. For a function $f : Y \to X$ defined on a set E in Y, f is a *representative function* of F on E if $f(y) \in F(y) = g^{-1}(y)$ for all $y \in E$. Note that this definition is equivalent to saying $g(f(y)) = y$ for all $y \in E$, which is also equivalent to saying that f is one-to-one on E with inverse $g : f(E) \to E$ (i.e., f is an inverse for g on E).

Example. Suppose that $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a branch of log on a domain $\mathcal{U}$ in $\mathbb{C} \setminus \{0\}$, so f is continuous on $\mathcal{U}$ and $f(z) \in \log(z)$ for all $z \in \mathcal{U}$ (by definition). Hence $e^{f(z)} = z$ for all $z \in \mathcal{U}$, and f is continuous on $\mathcal{U}$. Thus f is continuous on $\mathcal{U}$ and $f : \mathcal{U} \to f(\mathcal{U})$ is the inverse of $\exp : f(\mathcal{U}) \to \mathcal{U}$.

Example. For any $n \in \mathbb{N}$, $z^n : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is not one-to-one since its inverse $z^{1/n}$ is the multiple-valued inverse.

7. January 21: Sequences and series in $\mathbb{C}$

Definition 7.1. Let $\{a_n : n \geq 1\} \subset \mathbb{C}$ be a sequence in $\mathbb{C}$. Then $\{a_n\}$ *converges to* a if for any $\varepsilon > 0$ there exists $N > 0$ such that $|a_n - a| < \varepsilon$ (i.e., $a_n \in D(a, \varepsilon)$) for all $n \geq N$. We write $a_n \to a$ or $\lim a_n = a$.

Definition 7.2. Let $\{a_n\} \subset \mathbb{C}$. Then $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots, \sum a_j, \cdots$ forms a sequence in $\mathbb{C}$. If this sequence of partial sums converges, we say that *the series* $\sum_{n=0}^{\infty} a_n$ *converges*, and we define its sums to be

$$\sum_{n=0}^{\infty} a_n := \lim_{n\to\infty} \sum_{j=0}^{n} a_j.$$

Thus $\sum a_n$ denotes a complex number which we call the *sum of the series*. If the sequence of partial sums diverges then we say $\sum a_n$ *diverges*.

Definition 7.3. $\sum_{n=0}^{\infty} a_n$ *converges absolutely* if $\sum_{n=0}^{\infty} |a_n|$ converges as well.

Some results are in the handout, so refer to those. A few remarks regarding them:

Remark. The n-th term test is to test the *divergence* of a series. If $|a_n| \not\to 0$, then $\sum a_n$ diverges. Absolute convergence implies convergence, since

$$\left|\sum_{n=0}^{\infty} a_n\right| \leq \sum_{n=0}^{\infty} |a_n|,$$

which may be viewed as the infinite triangle inequality.

Proposition 7.1. *Suppose that $\sum a_n$ and $\sum b_n$ are convergent.*

(i) (sum) $\sum(a_n + b_n) = \sum a_n + \sum b_n$

(ii) (scalar product) If $\lambda \in \mathbb{C}$, then $\sum \lambda a_n = \lambda \sum a_n$. Thus if λ is a non-zero scalar, then $\sum a_n$ converges if and only if $\sum \lambda a_n$ converges.

(iii) (product of the series) in the handout.

Other results in handout.

Definition 7.4. Let $\{b_n\}$ be a sequence of real numbers. Then $\limsup b_n = \lim_{k\to\infty} \sup_{n\geq k} b_n$. In essence, $\limsup b_n$ is the largest y value such that $b_n \leq y$ for all $n \in \mathbb{N} \cup \{0\}$.

Remark. Visually, we can plot the sequence $\{b_n\}$, where the x-axis denotes n and the y-axis b_n. Then try to place $y = k$ so that none of the b_n's are above the graph, but once you lower it, there are bound to be some points above that line. This is one way to visualize lim sup.

Remark. $\limsup b_n$ always exists; if $\lim b_n$ exists, then $\lim b_n = \limsup b_n$.

Proposition 7.2 (Root test)**.** *Let $\sum a_n$ be a complex series. Then:*

(i) if $\limsup |a_n|^{1/n} < 1$ as $n \to \infty$ then the series converges absolutely.

(ii) if $\limsup |a_n|^{1/n} > 1$ or ∞ as $n \to \infty$, then the series diverges.

Both the ratio test and the root test can be proved as in $\mathbb{R}$, by comparison of the absolute value series with the geometric series.

Definition 7.5. For any $z \in \mathbb{C}$, the *geometric series with ratio z* is

$$\sum_{n=0}^{\infty} z^n.$$

If $|z| \geq 1$, then the series diverges by the n-th term test. Note that $\lim |z^n| = \lim |z|^n$ is either ∞ (if $|z| > 1$) or 1 (if $|z| = 1$). If $|z| < 1$, then by the partial sum method (same as in $\mathbb{R}$), we have $(1-z)S_n = 1 - z^{n+1}$ where $S_n := 1 + z + \cdots + z^n$.

Since $|z| < 1$, we have $z \neq$. Thus

$$S_n = \frac{1 - z^{n+1}}{1 - z}.$$

But $\lim |z^{n+1}| = \lim |z|^{n+1}| = 0$ since $|z| < 1$. Thus $\lim z^{n+1} = 0$, from which it follows that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}.$$

Therefore, in the unit disc $D(0, 1)$, $\sum z^n$ always converges, and

$$\sum_{n=0}^{\infty} \frac{1}{1 - z}.$$

Outside $D(0,1)$, $\sum z^n$ diverges despite $\frac{1}{1-z}$ still defined except when $z = 1$.

7.1. **Exponential series**

Recall that for $t \in \mathbb{R}$, the series

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

converges absolutely. For $z \in \mathbb{C}$, consider

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Observe that

$$\left| \frac{z^n}{n!} \right| = \frac{|z|^n}{n!},$$

that $|z| \in \mathbb{R}$, and that

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$$

(as usual in $\mathbb{R}$). Then $\sum \frac{z^n}{n!}$ converges absolutely, and we have

$$\left| \sum_{n=0}^{\infty} \frac{z^n}{n!} \right| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}.$$

We define $\operatorname{Exp} : \mathbb{C} \to \mathbb{C}$ (we will use Exp rather than exp since we don't know at this point if the complex exponential is indeed equal to the analogous version in $\mathbb{C}$)

$$\operatorname{Exp} := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Then $|\operatorname{Exp} z| \leq e^{|z|}$; and when $z = x \in \mathbb{R}$, then

$$\operatorname{Exp}(x) = \exp z = e^x.$$

Thus Exp = exp.

Similarly, we can define sin and cos as we did over $\mathbb{R}$, i.e., $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are complex-valued functions satisfying the following:

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}.$$

The geometric series, $\exp, \sin$, and cos are all examples of power series.

Definition 7.6. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $\{a_n\}$ is a complex-valued sequence.

Remark. For $z = 0$ the series is just equal to $a_0 + 0 + 0 + \cdots = a_0$, which absolutely converges, with a_0 the first term. If $z \neq 0$, then by the root test

$$\limsup |a_n z^n|^{1/n} = \limsup |a_n|^{1/n} |z| = |z| \limsup |a_n|^{1/n} = |z| l,$$

where $l := \limsup |a_n|^{1/n}$.

If $l = 0$ then $|z| l = 0 < 1$, so the series converges absolutely at each $z \in \mathbb{C}$. Thus the disk of convergence is the entire $\mathbb{C}$, and the radius of convergence is ∞.

If $l = \infty$ then $|z| l = \infty > 1$, so the series diverges as long as $z \neq 0$. Hence the disk of convergence is $\{0\}$, and the radius of convergence is 0.

If $0 < l < \infty$, then $|z| l < 1$ if and only if $|z| < l^{-1}$ (i.e., $|z| \in (0, l^{-1})$); similarly, $|z| > 1$ if and only if $|z| > l^{-1}$ (i.e., $z \notin \overline{D(0, l^{-1})}$). Therefore, the series converges absolutely in $D(0, l^{-1})$ and diverges for any $|z| > l^{-1}$. Hence the disk of convergence is $D(0, l^{-1})$, and the radius of convergence is l^{-1}. If $|z| = l^{-1}$, the test is inconclusive.

One can also apply the ratio test instead to find the radius of convergence. For $z \neq 0$, if a_n's don't converge eventually to 0, then

$$\lim_{n\to\infty} \frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} |z| = |z| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

Letting $\lim |a_{n+1}|/|a_n| = L$, we see that the radius of convergence is $L^{-1} = l^{-1}$. In conclusion, we have

$$R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{\lim_{n\to\infty} |a_{n+1}|/|a_n|}. \tag{2}$$

Definition 7.7. (2) is called *Hadamard's formula.*

What are some take-away messages from this section? Every power series $\sum a_n z^n$ has a radius of convergence $R = [0, \infty]$, and has a disk of convergence:

(i) $\{0\}$ if $R = 0$
(ii) $\mathbb{C}$ if $R = \infty$
(iii) $D(0, R)$ if $0 < R < \infty$.

The series converges absolutely at each z in its disk of convergence, and this defines a function

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}$$

for each $z \in D_{\text{conv}}$, where D_{conv} denotes the disk of convergence. The series diverges for any $|z| > R$.

In general, $f(z) = \sum a_n z^n$ means f is the function defined by the power series insider the disk of convergence. Thus we call 0 the *centre of the power series.*

Remark. Suppose that we know $\sum a_n z^n$ converges at some z. Then we know that z must be in the closure of the disk of convergence. Thus $R \geq |z|$. If $\sum a_n z^n$ diverges for some z, then $R \leq |z|$.

8. January 23

8.1. **Power series centred at $z_0 \in \mathbb{C}$**

Consider the series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

So in this case, the centre is z_0 rather than 0. Now fix $z \in \mathbb{C}$ and write $w := z - z_0$. Then

$$\sum_{n=0}^{\infty} = a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n w^n.$$

Thus the power series $\sum a_n (z - z_0)^n$ converges at z if and only if $\sum a_n z^n$ converges at $w = z - z_0$. The same equivalence holds for absolute convergence as well. So $g(z) = \sum a_n (z - z_0)^n$ has the *same* radius of convergence R, as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and its disk is just translated by z_0. All in all, we have, for $g(z) = f(z - z_0)$, the disk of convergence for $g(z)$ is $\{z_0\}$ if $R = 0$, $\mathbb{C}$ if $R = \infty$, and $D(z_0, R$ if $0 < R < \infty$.

8.2. **Basic properties of power series**

Definition 8.1. Let

$$\sum_{n=0}^{\infty} c_n = c_0 + c_1 + \cdots + c_{k-1} + \sum_{n=k}^{\infty} c_n.$$

Then a *tail* of the power series is the summand $\sum_{n=k}^{\infty} c_n$.

Remark. It is straightforward from the definition that the series converges if and only if any tail of a power series converges. If the given series converges, then the tails tends to 0 as $k \to \infty$. Indeed, note that

$$\sum_{n=0}^{\infty} a_n z^n = \underbrace{a_0 + a_1 z + \cdots + a_{k-1} z^{k-1}}_{\text{polynomial}} + \sum_{n=k}^{\infty} a_n z^n,$$

and so they have the same radius of convergence R and hence the same disk of convergence R also; and for any z in the disk of convergence, we have

$$\lim_{k \to \infty} \sum_{n=k}^{\infty} a_n z^n = 0.$$

Recall that for any λ,

$$\lambda \sum c_n = \sum \lambda c_n,$$

and that the LHS converges if and only if the RHS does, provided $\lambda \neq 0$.

So for some fixed $z \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$, we see that

$$z^k \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{n+k},$$

and so the LHS converges if and only if the RHS converges. Hence both have the same radius of convergence and the same disk of convergence. Hence for any power series $f(z)$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0+a_1z+\cdots+a_{k-1}z^{k-1}+sum_{n\geq k}a_n z^n = a_0+\cdots+a_{k-1}z^{k-1}+z^k\sum_{n=0}^{\infty} a_n z^{n-k}.$$

Note that both $\sum_{n\geq k} a_n z^n$ and $\sum_{n\geq k} a_n z^{n-k}$ both have the same R, and the same goes for $\sum a_n z^n$. In particular,

$$f(z) - f(0) = f(z) - a_0 = \sum_{n\geq 1} a_n z^n = z\sum_{n\geq 0} a_n z^{n-1},$$

which has the same radius R. Using this as a springboard, we will examine the continuity of $f(z)$ at the centre. Both $f(z)$ and $\sum_{n\geq 1} a_n z^{n-1}$ have the same R, so $f(z)$ converges absolutely for all $|z| \leq r < R/2$. So for all $|z| < r$,

$$\begin{aligned} |f(z) - f(0)| &= |z| \left|\sum_{n=1}^{\infty} a_n z^{n-1}\right| \\ &\leq |z| \sum_{n=1}^{\infty} |a_n||z|^{n-1} = C|z|, \end{aligned}$$

where

$$C := \sum_{n=1}^{\infty} |a_n| r^{n-1}.$$

Hence $|f(z) - f(0)| \leq C|z|$, which implies that f is continuous at 0. It follows that every power series is continuous at its centre.

For any series, we can shift the index, i.e.,

$$\sum_{n=0}^{\infty} c_n = c_0 + c_1 + \cdots = \sum_{n=k}^{\infty} c_{n-k}.$$

Indeed, $\sum c_{n-k}$ converges if and only if $\sum c_n$ does. We can do something similar for power series.

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + \cdots = \sum_{m=k}^{\infty} a_{m-k} z^{m-k}.$$

We can apply this concept to the tail of a power series.

$$\sum_{n=k}^{\infty} a_n z^n = z^k \sum_{n=k}^{\infty} a_n z^{n-k} = z^k \sum_{m=0}^{\infty} a_{m+k} z^m.$$

Note that all three series have the same radius, and that the second and the third series are the identical series.

8.2.1. *Sums and products.* For series, addition is defined by piece-wise addition, i.e., $\sum(c_n + d_n) = \sum c_n + \sum d_n$, provided both converge. As for the product,

Definition 8.2. Suppose that $\sum c_n$ and $\sum d_n$ converge, and suppose that one of them converges absolutely. Then multiplying the two series gives

$$\left(\sum c_n\right)\left(\sum d_n\right) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} c_j d_{n-j}.$$

This is called the *Cauchy product.*

Now, suppose that $\sum a_n z^n$ has radius $R_a > 0$ and $\sum b_n z^n$ has the radius $R_b > 0$. Then both $\sum(a_n + b_n)z^n$ and $\sum c_n z^n$ have radii of convergence $\min(R_a, R_b) =: R$, where $c_n = \sum_{j=0}^{n} a_j b_{n-j} = \sum_{i_1+i_2=n} a_{i_1} b_{i_2}$.

Remark. We will further examine some properties of the Cauchy product. If $a_0 = a_1 = \cdots = a_{k-1} = 0$ and $b_0 = b_1 = \cdots = b_{l-1} = 0$ but $a_k, b_l \neq 0$, then $c_0 = c_1 = \cdots = c_{k+l-1} = 0$, but $c_{k+l} \neq 0$ since $a_k b_l \neq 0$. Hence

$$\left(\sum_{n=k}^{\infty} a_n z^n\right)\left(\sum_{n=l}^{\infty} b_n z^n\right) = \sum_{n=k+l}^{\infty} c_n z^n.$$

Example. If $R_a \neq R_B$ then $\sum(a_n + b_n)z^n$ has radius $\min(R_a, R_b)$.

Example. $\cos z + i \sin z = \operatorname{Exp}(iz)$ for all $z \in \mathbb{C}$, so it is not that hard to verify via power series that

$$(\operatorname{Exp} z)(\operatorname{Exp} w) = \operatorname{Exp}(z + w)$$

for all $z, w \in \mathbb{C}$. In particular, $\operatorname{Exp} z = \operatorname{Exp}(x+iy) = (\operatorname{Exp} x)(\operatorname{Exp} iy) = e^x(\cos y + i \sin y) = e^x e^{iy} = \exp(z)$ for all $z \in \mathbb{C}$, giving us an alternate way of proving that $\operatorname{Exp} \equiv \exp$.

9. January 25

Let $f(z) = \sum a_n z^n$ have radius of convergence $R > 0$. Then for $|z| < R$, we have

$$f(z)^2 = (\sum a_n z^n)(\sum a_n z^n) = \sum c_{2,n} z^n,$$

where

$$c_{2,n} := \sum_{i_1+i+2=n} a_{i_1} a_{i_2},$$

and

$$f(z)^2 = \sum_{n\geq 0} c_{2,n} z^n$$

has radius $\geq R$. But then for $|z| < R$,

$$f(z)^3 = \left(\sum_{n\geq 0} c_{2,n} z^n\right)\left(\sum_{n\geq 0} a_n z^n\right) = \sum_{n\geq 0} c_{3,n} z^n$$

with

$$c_{3,n} = \sum_{i+i_3=n} c_{2,i} a_{i,3} = \sum_{i_1+i_2+i_3=n} a_{i_1} a_{i_2} a_{i_3}.$$

So by induction, for $|z| < R$

$$f(z)^k = \left(\sum_{n=0}^{\infty} a_n z^n\right)^k = \sum_{n=0}^{\infty} c_{k,n} z^n,$$

and

$$c_{k,n} = \sum_{i_1+\cdots+i_k=n} a_{i_1} a_{i_2} \cdots a_{i_k}.$$

Notice that if the constant term $a_0 = 0$, then

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

with $R > 0$. Then $f(z)^k$ starts at z^k in this case.

9.1. **Composition of power series**

Let $\sum_{n\geq 1} a_n z^n$ and $\sum_{n\geq 0} b_n z^n$ with radius $R_a > 0$ and $R_b > 0$ respectively. Then f is continuous at 0 and $f(0) = 0$, This means there exists $\delta > 0$ such that $\delta < R_a$ and $|f(z)| < R_b$ for $|z| < \delta$. So for $|z| < delta$, $f(z)$ is defined and is inside the disk of convergence for g. Note that

$$\begin{aligned} g(f(z)) &= \sum_{k=0}^{\infty} b_k f(z)^k = b_0 + \sum_{k=1}^{\infty} b_k f(z)^k \\ &= b_0 + \sum_{k=1}^{\infty} b_k \sum_{n=k} c_{k,n} z_n. \end{aligned}$$

One natural question now arises: can we switch the two summands? See the notes distributed during the class for more information on this. Thus we have

$$\begin{aligned} b_0 &= b_0 \\ b_1 f(z) &= \qquad\qquad + b_1 c_{11} z + b_1 c_{12} z^2 + b_1 c_{13} z^3 + \cdots \\ b_2 f(z)^2 &= \qquad\qquad\qquad\qquad + b_2 c_{22} z^2 + b_2 c_{23} z^3 + \cdots \end{aligned}$$

Since we can change the summands,

$$\sum_{k=1}^{\infty} \sum_{n=k}^{\infty} |w_{k,n}| = \sum_{n=1}^{\infty} \sum_{k=1}^{n} |w_{k.n}|,$$

and they obviously have the same radii of convergence. So we need convergence of

$$\sum_{k=1}^{\infty} |b_k| \sum_{n=k}^{\infty} |c_{k,n}||z|^n,$$

where $c_{k,n} = \sum_{i_1+\cdots+i_k=n} a_{i_1}a_{i_2}\cdots a_{i_k}$. But then

$$\sum_{k\geq 1} |b_k| \sum_{n\geq k} |c_{k,n}||z|^n \leq \sum_{k=1}^{\infty} |b_k| \sum_{k\geq k} C_{k,n}|z|^n,$$

where $C_{k,n} := \sum_{i_1+\cdots+i_k=n} |a_{i_1}|\cdots|a_{i_k}|$. Hence

$$\sum_{k=1}^{\infty} |b_k| \sum_{k\geq k} C_{k,n}|z|^n = \sum_{k\geq 1} |b_k| \left(\sum_{n=1}^{\infty} |a_n||z|^k\right)^k = G(F(z)),$$

where $G(z) = \sum |b_k|z^k$ with the same radius of convergence R_b (same as g) and $F(z) = \sum_{n\geq 1} |a_n|z^n$ with the same radius of convergence R_a (same as f).

Since $F(|\cdot|) = F \circ |\cdot|$ is continuous at 0 and $F(0) = 0$, there is $r > 0$ such that $r < R_a$ and $F(|z|) < R$ whenever $|z| < r$. So

$$\sum_{k\geq 1} |b_k| \left(\sum_{n\geq 1} |a_n||z|^k\right)^k$$

converges, which means we can switch the two summands. Furthermore, $|z| < R_a$ so $\sum a_n z^n$ converges; also, $|f(z)| = |\sum a_n z^n| \leq \sum |a_n||z|^n = F(|z|) < R_b$. Hence

$$\begin{aligned} g(f(z)) &= b_0 + \sum_{k=1}^{\infty} b_k \left(\sum_{n=1}^{\infty} a_n z^n\right)^k \\ &= b_0 + \sum_{k=1}^{\infty} b_k \sum_{n=k}^{\infty} c_{k,n} z^n \\ &= b_0 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} b_k c_{k,n}\right) z^n. \end{aligned}$$

Note that the change of summands works because $\sum b_k (\sum a_n z^n)^k$ converges. In conclusion we have the following theorem.

Theorem 9.1. *Let*

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} b_n z^n \\ f(z) &= \sum_{n=1}^{\infty} a_n z^n, \end{aligned}$$

whose radii of convergence are R_b and R_a respectively. If $0 < r < R_a$ such that (note that such an r exists)

$$\sum_{n=1}^{\infty} |a_n| r^n < R_b,$$

then $g(f(z))$ converges for $|z| < r$. If $R_b = \infty$, then for any $r < R_a$, the composition $g(f(z))$ converges for all $|z| < r$. Hence $g(f(z))$ converges for all $|z| < R_a$.

9.2. **Reciprocals**

Let $f(z) = a_n z^n$ with radius of convergence $R > 0$. What can we say about $1/f(z)$? Clearly we need $f(0) \neq 0$ i.e., constant term cannot be 0. Assume that $a_0 = 1$, since we can always multiply by a constant for general case. Notice that if

$$f(z) = 1 + \underbrace{\sum_{n=1}^{\infty} a_n z^n}_{\text{has the same radius } R}$$

then

$$\frac{1}{f(z)} = \frac{1}{1-(1-f(z))} = \sum_{n=0}^{\infty}(1-f(z))^n,$$

provided that $|1-f(z)| < 1$ (geometric series). Using composition,

$$g(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

has radius 1, and

$$h(z) = 1 - f(z) = -\sum_{n=1}^{\infty} a_n z^n$$

has the radius R. So there exists r such that $0 < r < R$ so that $|h(z)| = |1-f(z)| < 1$ and $\frac{1}{f(z)} = g(h(z))$ is convergens for $|z| < r$. All in all we proved the following theorem.

Theorem 9.2. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with $a_0 \neq 0$ and radius $R > 0$. Then there exists $r > 0$ such that $1/f(z)$ converges for all $|z| < r$.

10. January 28

Is f holomorphic in $|z| < R$ with $f'(z) = g(z)$? That is, for $|w| < R$ we want

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} \stackrel{?}{=} g(w).$$

For each z and w in the disk of convergence, we know that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + \sum_{n=1}^{\infty} a_n z^n$$

$$f(w) = \sum_{n=0}^{\infty} a_n w^n = a_0 + \sum_{n=1}^{\infty} a_n w^n$$

$$g(w) = \sum_{n=1}^{\infty} n a_n w^{n-1}.$$

Note that all three series are convergent series of complex numbers, so must have some radius of convergence.

For any $z \neq w$ with z in this disk of convergence, we have

$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| = \left|\sum_{n=1}^{\infty} a_n \left[\frac{z^n - w^n}{z-w} - nw^{n-1}\right]\right| \leq \sum_{n=2}^{\infty} |a_n| \underbrace{\left|\frac{z^n - 2^n}{z-w} - nw^{n-1}\right|}_{0 \text{ if } n=1}$$

But then from the first assignment, we know

$$\left|\frac{z^n - w^n}{z-w} - nw^{n-1}\right| = \left|(z-w)\sum_{j=1}^{n-1} jz^{(n-1)-j}w^{j-1}\right| \leq |z-w| \sum_{j=1}^{n-1} j|z|^{(n-1)-j}|w|^{j-1}.$$

If both $|z|, |w| < r$ then we have

$$|z-w|\sum_{j=1}^{n-1} j|z|^{(n-1)-j}|w|^{j-1} < |z-w| \underbrace{\sum_{j=1}^{n-1} j}_{\frac{n(n-1)}{2} \leq n^2} \; r^{n-2} \leq |z-w|n^2 r^{n-2}.$$

Now let $|z|, |w| < r < R$ and $z \neq w$. Then z, w are both in the dis kof convergence, so there is a constant $C > 0$ such that

$$0 \leq \left|\frac{f(z)-f(w)}{z-w} - g(w)\right| \leq \sum_{n=2}^{\infty} |a_n||z-w|n^2 r^{n-2} \leq C|z-w| = |z-w|\sum_{n=2}^{\infty} |a_n| n^2 r^{n-2}.$$

Because $r < R$, by the root test, we see that $\sum_{n=2}^{\infty} |a_n| n^2 r^{n-2}$ converges by the root test:

$$\limsup_n (|a_n| r^{n-2})^{1/n} = r \limsup_n |a_n|^{1/n} = rR^{-1} < 1.$$

So by the squeeze theorem as $z \to w$, we see that

$$0 \leq \left|\frac{f(z)-f(w)}{z-w} - g(w)\right| \leq \sum_{n=2}^{\infty} |a_n||z-w|n^2 r^{n-2} \leq 0.$$

Hence

$$\lim_{z \to w} \frac{f(z)-f(w)}{z-w} = g(w)$$

for all $|w| < R$. This proves the following theorem.

Theorem 10.1. *If $\sum a_n z^n$ have radius of convergence $R > 0$, then the function defined by $f(z) := \sum a_n z^n$ is holomorphic on $|z| < R$ with*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

i.e., term-by-term differentiation of each term. Furthermore, $f'(z)$ has the same radius of convergence R.

Corollary 10.1. *If $\sum a_n(z-z_0)^n$ has radius of convergence $R > 0$, then $F(z)$ is holomorphic on $|z - z_0| < R$ with*

$$F'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^n,$$

and both $F(z)$ and $F'(z)$ have the same radius of convergence R.

Proof. Let $F(z) = f(z - z_0)$. Then $F'(z) = f'(z - z_0)(z - z_0)' = f'(z - z_0)$ by the chain rule. $\square$

So we show that

$$F'(z) = \sum n a_n (z - z_0)^{n-1}$$

with the same radius of convergence $R > 0$, then F' is holomorphic on $|z - z_0| < R$ with

$$F''(z) = \sum_{n=2}^{\infty} n(n-1) a_n (z - z_0)^{n-2}$$

with the same R. Now applying induction onto this argument, we prove the following corollary.

Corollary 10.2. *If $a_n(z - z_0)^n$ has radius $R > 0$, then $f(z) = \sum a_n(z - z_0)^n$ is infinitely complex differentiable on $|z - z_0| < R$ with*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z - z_0)^{n-k} = k! \sum_{n=k}^{\infty} a_n \binom{n}{k} (z - z_0)^{n-k}$$

with the constant term $k! a_k$; and for $k \in \mathbb{N}$, all have the same radius of convergence R. So in fact,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

i.e., the Taylor series of f at $z = z_0$.

Finally, we remark on the uniqueness of such power series. Suppose that $\sum a_n(z-z_0)^n$ and $\sum b_n(z-z_0)^n$ represent the same function with R_a, R_b each radius of convergence respectively. Suppose $R < R_a, R_b$. Then $a_n = b_n$ for all n.

Definition 11.1. Given $f(z) = \sum a_n z^n$ with radius of convergence $R > 0$, the *primitive power series* of $f(z)$ is just a term-by-term integral, i.e.,

$$g(z) = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1} = a_0 z + \frac{a_1 z^2}{2} + \cdots$$

is the primitive power series of $f(z)$.

Both series have the same radius R, and $g'(z) = f(z)$ for $|z| < R$. Thus, g is a primitive for f on $|z| < R$.

Definition 11.2. Let $f : \mathbb{C} \to \mathbb{C}$ defined on an open set $\mathcal{U}$. Then f has a *power series expansion* at $z_0 \in \mathcal{U}$ if there exists a power series $\sum a_n(z - z_0)^n$ centred at z_0 with radius of convergence $R > 0$ so that $f(z) = \sum a_n(z - z_0)^n$ for all z in some disk around z_0.

Remark. If f has a power series expansion at z_0, then f is infinitely differentiable in a disk around z_0. Also, there is only one possibility for the power series. Therefore, this power series matches the Taylor series; that is,

$$a_n = \frac{f^{(n)}(z_0)}{n!} \text{ for all } n.$$

Definition 11.3. f is *analytic* in $\mathcal{U}$ if f has a power series expansion at each $z_0 \in \mathcal{U}$.

Remark. Every power series expansion defines an analytic functions in its disk of convergence.

Proposition 11.1. *If f is analytic on $\mathcal{U}$, then f is also holomorphic on $\mathcal{U}$; furthermore, all derivatives are analytic. Therefore, f has a local primitive in $\mathcal{U}$ (i.e., for each $z \in \mathcal{U}$ there exists a disk around z on which f has a primitive or a power series expansion at z_0).*

Proposition 11.2. *If f is analytic on $\mathcal{U}$ and g is the primitive of f on $\mathcal{U}$, then g is also analytic on $\mathcal{U}$.*

Proposition 11.3. *If f and g are analytic on $\mathcal{U}$, then so are $f + g$ and fg. f/g is analytic on $\mathcal{U} \setminus \{g = 0\}$. If $f : \mathcal{U} \to \mathcal{V}$ and $g : \mathcal{V} \to \mathbb{C}$ where $f(\mathcal{U}) \subseteq \mathcal{V}$, and both f and g are analytic on $\mathcal{U}$ and $\mathcal{V}$ respectively, then $g \circ f$ is analytic on $\mathcal{U}$.*

11.1. **Isolation at zero**

Let f be an analytic function on domain $\mathcal{U}$. Define

$$Z := \{\text{zeros of } f\} = \{z \in \mathcal{U} : f(z) = 0\} =: f^{-1}(0).$$

Let $z_0 \in Z$. Then there is a power series expansion for $f(z)$ at z_0 on some disk $D(z_0, \delta)$. But recall that either f is either 0 on $D(z_0, \delta)$ (i.e., $z_0 \in Z^\circ$) or $f \neq 0$ on some $D^*(z_0, \delta)$ so $z_0 \in Z_{\text{iso}}$. If $z_0 \in \mathcal{U} \setminus Z$ so that $f(z_0) \neq 0$, then $f \neq 0$ on some $D(z_0, \delta)$ as f is continuous at z_0. So $D(z_0, \delta) \in Z^c$ so $z_0 \in Z_{\text{ext}}$. Hence $\mathcal{U} = Z \cup (\mathcal{U} \setminus Z) = Z^\circ \cup Z_{\text{iso}} \cup (\mathcal{U} \cap Z_{\text{ext}})$. Note that Z° is open by definition; so is $Z_{\text{iso}} \cup (U \cap Z_{\text{ext}}) = Z_{\text{iso}} \cup Z_{\text{ext}}$ since it is the union of two open sets. Also Z° and $Z_{\text{iso}} \cup Z_{\text{ext}}$ are disjoint, making them separated sets. But then $\mathcal{U}$ is connected and open (since $\mathcal{U}$ is a domain), so either $\mathcal{U} = Z^\circ$ or $\mathcal{U} \subset Z_{\text{iso}} \cup Z_{\text{ext}} = (Z')^c$. In conclusion, either $f \equiv 0$ on $\mathcal{U}$ (if $\mathcal{U} = Z^\circ$) or the zeroes of f are isolated, and do not accumulate in $\mathcal{U}$ (if $\mathcal{U} \subset Z_{\text{iso}} \cup Z_{\text{ext}}$). However, since $Z' = (Z_{\text{iso}} \cup Z_{\text{ext}})^c \subset \mathcal{U}^c$, the zeros can still accumulate in $\partial\mathcal{U}$, the boundary of $\mathcal{U}$.

Definition 12.1. Consider $h : [a, b] \to \mathbb{C}$ where $h(t) = x(t) + iy(t)$. Then h is *Riemann-integrable* if both $x, y : [a, b] \to \mathbb{R}$ are. In other words, if

$$\int_a^b h(t)\,dt := \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt,$$

then both $\int_a^b x(t)$ and $\int_a^b y(t)$ exist.

Proposition 12.1. *If $h : [a, b] \to \mathbb{C}$ is Riemann-integrable, then h has all the usual properties that Riemann-integrable real functions have.*

- $\displaystyle\int_a^b f + g = \int_a^b f + \int_a^b g$
- $\displaystyle\lambda\int_a^b f = \int_a^b \lambda f$ *(where $\lambda \in \mathbb{C}$*
- *If h in integrable, then so is $|h|$, and*

  $$\left|\int_a^b h\right| \le \int_a^b |h|.$$

  Note that $h : [a, b] \to \mathbb{C}$ whereas $|h| : [a, b] \to \mathbb{R}$.
- *If h is integrable on $[a, b]$ and $a \le c \le b$, then*

  $$\int_a^b h = \int_a^c h + \int_c^b h.$$

- *If h is continuous on $[a, b]$, then h is integrable.*
- *If g is integrable on $[a, b]$, and $h = g$ on $[a, b] \setminus F$ where F is a finite set, then $\int h$ can be defined even if h is undefined at finitely many points.*
- *If h is bounded on $[a, b]$ and is continuous on $[a, b] \setminus F$, then h is integrable.*

Additionally, h satisfies

- $\displaystyle\mathrm{Re}\int_a^b h(t)\,dt = \int_a^b \mathrm{Re}\,h(t)\,dt$
- $\displaystyle\mathrm{Im}\int_a^b h(t)\,dt = \int_a^b \mathrm{Im}\,h(t)\,dt$
- $\displaystyle\overline{\int_a^b h(t)\,dt} = \int_a^b \overline{h(t)}\,dt.$

Remark. From the above proposition, it follows that h is integrable if h is piecewise-continuous on $[a, b]$. Piecewise-continuity implies that h is continuous on $[a, b] \setminus F$ for some finite set F, and that both the left-sided limit and the right-sided limit exist at each $p \in F$.

We also have integration by substitution in complex analysis as well. Suppose that $u : [a, b] \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{C}$. If u is C^1 and h is continuous on $u([a, b])$, then

$$\int_{u(a)}^{u(b)} h(u)\,du = \int_a^b h(u(t))u'(t)\,dt.$$

In fact, the substitution rule can be used even when the conditions are slightly weakened.

Proposition 12.2 (Integration by substitution)**.** *Suppose* $u : [a, b] \to \mathbb{R}$ *is* piecewise-C^1 *on* $[a, b]$*, and* $h : \mathbb{R} \to \mathbb{C}$ piecewise-continuous *on* $u([a, b])$*. Furthermore, suppose that* $u^{-1}(F)$ *is a finite set, where* F *is the set of points where* h *is not continuous. Then*

$$\int_{u(a)}^{u(b)} h(u)\,du = \int_a^b h(u(t))u'(t)\,dt.$$

Proposition 12.3 (Integration by parts)**.** *Suppose* $g, h : [a, b] \to \mathbb{C}$ *which are both piecewise-*C^1*. Then*

$$\int_a^b gh' = gh|_a^b - \int_a^b h'g.$$

12.1. **Paths and curves in $\mathbb{C}$**

Definition 12.2. A *path* is any continuous map γ from $[a, b] \subset \mathbb{R}$ to $\mathbb{C}$. We can also add adjectives such as C^1, differentiable, smooth, piecewise-smooth, and piecewise-C^1 depending on which additional properties are satisfied. A path $\gamma : [a, b] \to \mathbb{C}$ is *closed* if $\gamma(a) = \gamma(b)$. A path is *simple* if $\gamma(t) = \gamma(s)$ implies $t = s$ or $\{t, s\} = \{a, b\}$.

One can also join two paths. Suppose that $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are paths, and that γ_1 ends where γ_2 starts. Then $\gamma_1 + \gamma_2 : [a, b + d - c] \to \mathbb{C}$ is a path where

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t + c - b) & t \in [b, b + d - c]. \end{cases}$$

Definition 12.3. The *opposite path* of $\gamma : [a, b] \to \mathbb{C}$ is $\gamma^- : [a, b] \to \mathbb{C}$ such that $\gamma^-(t) = \gamma(a + b - t)$ for $t \in [a, b]$.

Definition 12.4. Given a path $\gamma_1 : [a, b] \to \mathbb{C}$, γ_2 is a *reparametrization* of γ_1 if $\gamma_2 : [c, d] \to \mathbb{C}$ satisfies $\gamma_2(s) = \gamma_1(t(s))$ where $t : [c, d] \to [a, b]$ is an increasing homeomorphism.

Remark. The reparametrization relation is an equivalence relation on the set of all paths. In other words, if γ_2 is a reparametrization of γ_1, then $\gamma_1 \equiv \gamma_2$, and that $\equiv$ satisfies all the conditions for an equivalence relation.

Definition 12.5. A *curve* is an equivalence class of paths.

13. February 8 & 11

Consider the $\overline{zw}$ curve parametrized by $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(t) = z + t(w - z)$. Then $\gamma'(t) = w - z$.

Definition 13.1.

Definition 13.2. Suppose $a \in C$ and $r > 0$. Then $C = C_r = C_r(a)$ is the smooth curve with parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$ such that $t \mapsto a + re^{it}$. $C_r(a)$ denotes the circle, and $\gamma'(t) = ire^{it}$. We can also parametrize the same curve in the following way: $\gamma : [0, 1] \to \mathbb{C}$ so that $t \mapsto a + re^{2\pi it}$. In this case, $\gamma'(t) = 2\pi ire^{(2\pi i)t}$. This orientation is the *positive orientation* for the circle (counterclockwise). The opposite curve C^- is called the *negative orientation.* A segment of the circle is called an *arc.*

Definition 13.3. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise-C^1 path, and $f : \mathbb{C} \to \mathbb{C}$ continuous on $\gamma([a,b])$. Then let

$$\int_\gamma f := \int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

is the *line integral* of f along the curve γ.

Remark. Notice that $h(t) := f(\gamma(t))\gamma'(t)$ is piecewise-continuous on $[a,b]$. So the Riemann integral $\displaystyle\int_a^b h(t)\,dt$ is well-defined.

Suppose that $t : [c,d] \to [a,b]$ a piecewise-smooth increasing diffeomorphism. If $\widetilde{\gamma} := \gamma \circ t$, then

$$\begin{aligned}\int_{\widetilde{\gamma}} f &= \int_c^d f(\widetilde{\gamma}(s))\widetilde{\gamma}'(s)\,ds \\ &= \int_c^d f(\gamma(t(s))\gamma'(t(s))t'(s)\,ds.\end{aligned}$$

Let $f(\gamma(t(s)))\gamma'(t(s)) =: h((t(s)))$, where $h : [a,b] \to C$ is piecewise-continuous, and $t : [c,d] \to [a,b]$ is piecewise-C^1 and one-to-one. Then by substitution,

$$\begin{aligned}\int_c^d f(\gamma(t(s))\gamma'(t(s))t'(s)\,ds &= \int_{t(c)}^{t(d)} f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_\gamma f.\end{aligned}$$

From this, we see that reparatmetrization doesn't alter the integral. So if γ is a contour, we can define $\displaystyle\int_\gamma f$ can be defined unambiguously, since we get the same answer using any parametrizaton of γ.

Now suppose that a curve given is an opposite curve. Given $\gamma : [a,b] \to \mathbb{C}$ piecewise-C^1, consider $\gamma^- : [a,b] \to \mathbb{C}$ which is defined $s \mapsto \gamma(a+b-s)$. Hence let $t(s) = a+b-s$. Then t is a smooth diffeomorphism. Note that this is *not* a change of variables since $t(s)$ is decreasing. But we can make this into a change of variables via substitution as before.

Proposition 13.1 (Properties of the line integral)**.** *Suppose that γ_1, γ_2 are piecewise-C^1 paths. Then*

$$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

Definition 13.4. We often write

$$\oint_\gamma f = \int_\gamma f$$

whenever γ is a piecewise-C^1 closed curve, in order to emphasize that γ is closed.

13.1. **Invariance of starting point**

Let γ be piecewise-C^1 path that starts and ends at $p \in \mathbb{C}$. Given any q on γ, we have $\gamma = \gamma_1 + \gamma_2$ for piecewise-C^1 paths γ_1 from p to q and γ_2 from q to p. Then

$$\oint_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f = \int_{\gamma_2} f + \int_{\gamma_1} f = \oint_{\gamma_2+\gamma_1} f,$$

where $\gamma_2 + \gamma_1$ is the closed path starting and ending at q.

Linearity also holds for line integrals, i.e.,

$$\int_\gamma (\lambda f + g) = \lambda \int_\gamma f + \int_\gamma g.$$

13.2. **Arc length and integrals with respect to arc length**

Definition 13.5. The *arc length* of a piecewise-C^1 path $\gamma : [a, b] \to \mathbb{C}$ is

$$L(\gamma) := \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

where $\gamma(t) = x(t) + iy(t)$, $\gamma'(t) = x'(t) + iy'(t)$, and hence $|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.

Definition 13.6. Let f be continuous on γ. Then we define

$$\int_\gamma f\, |dz| := \int_a^b f(\gamma(t))|\gamma'(t)|\, dt = \int_a^b f(\gamma(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Notice in particular that

$$L(\gamma) = \int_\gamma |dz|.$$

Proposition 13.2. *Suppose that $\widetilde{\gamma}$ is a reparametrization of piecewise-smooth γ. Then*

$$\int_\gamma f\, |dz| = \int_{\widetilde{\gamma}} f\, |dz|.$$

Proposition 13.3 (Properties of arc length integrals)**.** *Suppose that γ_1, γ_2 are piecewise-C^1 paths.*

(1) (additivity) $\displaystyle\int_{\gamma_1+\gamma_2} f\, |dz| = \int_{\gamma_1} f\, |dz| + \int_{\gamma_2} f\, |dz|$

(2) (opposite curve) $\displaystyle\int_{\gamma^-} f\, |dz| = \int_\gamma f\, |dz|$

(3) (closed curves) $\displaystyle\oint_\gamma f\, |dz|$ *remains the same regardless of the starting point.*

Therefore, we have

(i) $L(\gamma) = L(\widetilde{\gamma})$ *whenever* $\widetilde{\gamma} \equiv \gamma$

(ii) $L(\gamma_1 + \gamma_2) = L(\gamma_1) + L(\gamma_2)$

(iii) $L(\gamma^-) = L(\gamma)$

(iv) $L(\gamma)$ *is invariant regardless of the starting point, provided that* γ *is a closed curve.*

Example. $L(C_r(a)) = 2\pi r$ regardless of the starting point.

13.3. **Bounding in $\int f$**

$$\begin{aligned}\left|\int_\gamma f\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\ &\le \int_a^b |f(\gamma(t))\gamma'(t)\,dt| \\ &= \int_a^b |f(\gamma(t))|\cdot|\gamma'(t)|\,dt.\end{aligned}$$

Proposition 13.4 (Additional properties of line integrals)**.** *Suppose f and g are continuous on γ, and that $\gamma:[a,b]\to\mathbb{C}$ is a piecewise-C^1 path.*

(i) (monotonicity) if $|f|\le|g|$, then

$$\int_\gamma |f|\,|dz| = \int_a^b |f(\gamma(t))|\cdot|\gamma'(t)|\,dt$$

$$\le \int_a^b |g(\gamma(t))|\cdot|\gamma'(t)|\,dt = \int_\gamma |g|\,|dz|.$$

In particular, if $|f|\le C$ on γ, then $\left|\int_\gamma f\right| \le C\int_\gamma |dz| = CL(\gamma)$.

13.4. **Fundamental theorem of calculus for contour integrals**

Theorem 13.1 (Fundamental theorem of calculus II)**.** *If $h:[a,b]\to\mathbb{C}$ is continuous with primitive H on $[a,b]$, then* $\int_a^b h = H(b)-H(a)$.

So for this, we need that H to be C^1 on $[a,b]$ and $H'=h$. Since $\int$ is additive, we in fact only need that H to be piecewise-C^1 on $[a,b]$ and $H'=h$.

Suppose that γ is a piecewise-C^1 path such that $\gamma(a)=p, \gamma(b)=q$ where $p,q\in\mathcal{U}$ fan open set. Suppose also that f is continuous on $\mathcal{U}$ with $F'=f$. Then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\ &= (F\circ\gamma)'(t)\,dt = (F\circ\gamma)(b)-(F\circ\gamma)(a) = F(q)-F(p).\end{aligned}$$

For a closed curve, since $p=q$, clearly

$$\int_\gamma f = F(p)-F(p) = 0.$$

So, for any continuous $\gamma:[a,b]\to\mathcal{U}$ we could define

$$\int_\gamma f = F(q)-F(p)$$

where q is the endpoint and p is the starting point. Since we have a primitive F for f on all of $\mathcal{U}$, now every derivative we know tells us what the integrals are.

Example. We know that for $n = 0, 1, 2, \ldots,$, the function z^{n+1} is entire with derivative $(n+1)z^n$. Therefore z^n has primitive $z^{n+1}/(n+1)$ on all of $\mathbb{C}$. So for any piecewise-C^1 path in $\mathbb{C}$,

$$\int_\gamma z^n\,dz = \frac{z^{n+1}}{n+1}\bigg|_p^q = \frac{q^{n+1}}{n+1} - \frac{p^{n+1}}{n+1}.$$

Furthermore, if γ is closed, then $\int z^n\,dz = 0$.

Example. For $n = 2, 3, \ldots,$ the function $z^{-(n-1)}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ with derivative $-(n-1)z^{-n}$. So for $n = 2, 3, \ldots,$ z^{-n} has primitive $-[(n-1)z^{n-1}]^{-1}$ on all of $\mathbb{C} \setminus \{0\}$. Indeed,

$$\int_\gamma \frac{dz}{z^n} = \frac{-1}{(n-1)z^{n-1}}\bigg|_p^q = \frac{-1}{(n-1)q^{n-1}} + \frac{1}{(n-1)p^{n-1}}.$$

If γ is closed, again we have $\int dz/z^n = 0$.

So far, we saw z^n where $n = 1, 2, 3, \cdots$ or $n = -2, -3, -4, \ldots$. What if $n = -1$? It has a primitive on each sliced plane $\mathbb{C} \setminus \mathrm{Ray}_\theta$. In fact, we need to be a bit cautious – suppose that we integrate z^{-1} along the circle, with γ_1 the path from angle 0 to $\pi/2$, and γ_2 from $\pi/2$ to 2π. Then in fact

$$\int_{\gamma_1} \frac{dz}{z} \neq \int_{\gamma_2} \frac{dz}{z}.$$

Theorem 13.2 (Fundamental theorem of calculus I)**.** *If $h : [a, b] \to \mathbb{C}$ continuous, then*

$$H(t) = \int_a^t h$$

is C^1 on $[a, b]$; furthermore, $H' = h$. Also, by additivity, if h is piecewise-continuous on $[a, b]$, then $H(t)$ is piecewise-C^1 on $[a, b]$ and $H' = h$.

Let $\mathcal{U}$ be an open set, with $p \in \mathcal{U}$ fixed. If f is continuous on $\mathcal{U}$, is it possible to define a primitive F on f by

$$F(z) = \int_\gamma f$$

where γ is piecewise-C^1 path from p to z in $\mathcal{U}$? For this definition to make sense, we would need a path γ in $\mathcal{U}$ from p to z; in other words we would want $\mathcal{U}$ to be connected (i.e., $\mathcal{U}$ is a domain). We also need the primitive to be well-defined; that is, we want the same answer γf for every path γ in $\mathcal{U}$ from p to z. This brings us the notion of path independence.

14. February 15

14.1. **Path independence**

Definition 14.1. We say that the integrals of f are *type-P path independent in $\mathcal{U}$* if for any $p, q \in \mathcal{U}$ and γ_1, γ_2 are type-P paths in $\mathcal{U}$ from p to q, then $\int_{\gamma_1} f = \int_{\gamma_2} f$.

We are defining the restrictions that need to be placed on paths for this definition to work; we will explore what restrictions we need as we go on.

Proposition 14.1. *$\int f$ is type-P path independent in $\mathcal{U}$ if and only if $\oint f = 0$ for all closed γ in $\mathcal{U}$ of type P.*

Note that we already know that the above proposition holds for certain types of paths:

(i) P_1: rectangular paths
(ii) P_2: polygonal paths
(iii) P_3: piecewise-smooth paths
(iv) P_4: piecewise-C^1 paths.

First, note that each type P above is closed under joining, i.e., for any joinable $\gamma_1, \gamma_2 \in P_i$, we have $\gamma_1 + \gamma_2 \in P_i$. We also have $P_1 \subset P_2 \subset P_3 \subset P_4$. We also saw from FTC II that if f is continuous on an open set $\mathcal{U}$, then not only does f have a primitive f on $\mathcal{U}$ (call it F), we also have

$$\int_\gamma f = F(q) - F(p)$$

where q is the ending point of the γ and p the starting point of γ (provided γ is piecewise-C^1). Hence the same claim holds for all type-P_4, so it holds for P_3, P_2, and P_1 paths also.

But what we are interested in is the *converse* of this. One reasonable hypothesis: perhaps the converse will hold if $\mathcal{U}$ is a domain. Notice that it suffices to prove the converse for all rectangular paths (since then the claim will follow for other types).

Suppose that f is continuous on domain $\mathcal{U}$, and that the integrals of f are rectangular-path-independent in $\mathcal{U}$. Start by fixing $p \in \mathcal{U}$. For all $z \in \mathcal{U}$, define

$$F(z) = \int_\gamma f$$

where γ is a rectangular path in $\mathcal{U}$ from p to z. Since $\mathcal{U}$ is a domain then it is rectangular-path-connected. Therefore γ exists. Also, F is well-defined: $\displaystyle\int_\gamma f$ has the same value for all rectangular paths γ in $\mathcal{U}$ from p to z. Finally, we need to show that F is holomorphic in $\mathcal{U}$ with $F' = f$. Let $a \in \mathcal{U}$. We need to show that

$$F'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = f(a).$$

Let γ be any rectangular path in $\mathcal{U}$ from p to a. Then

$$F(a) = \int_\gamma f.$$

Since $\mathcal{U}$ is open, there exists a disk $D(a, r) \subset \mathcal{U}$. Thus for all $z \in D(a, r)$, the path az lies in $D(a, r)$ so hence in $\mathcal{U}$. Thus $\gamma + az$ is a rectangular path in $\mathcal{U}$ from p to z. Thus

$$F(z) = \int_{\gamma + az} f = \int_\gamma f + \int_{az} f = F(a) + \int_{az} f.$$

And so for any $z \in D^*(a, r)$, we have

$$\frac{F(z) - F(a)}{z - a} = \frac{1}{z - a} \int_{az} f.$$

But recall that

$$\int_{az} dz = z - a.$$

So we can write
$$f(a) = \frac{1}{z-a}\int_{az} f(a)\,dz.$$
Putting these facts together, we have
$$\begin{aligned}\left|\frac{F(z)-F(a)}{z-a} - f(a)\right| &= \frac{1}{|z-a|}\left|\int_{az}(f(z)-f(a))\,dz\right| \\ &\le \frac{1}{|z-a|}\int_{az}|f(z)-f(a)|\,|dz|.\end{aligned}$$
Let's see what happens if $z \to 0$. Since f is continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ so that $D(a,\delta) \subset D(a,r)$ and $|f(z)-f(a)| < \varepsilon/2$ for all $z \in D(a,\delta)$. So for any $z \in D^*(a,\delta)$,
$$\left|\frac{F(z)-F(a)}{z-a} - f(a)\right| < \frac{1}{|z-a|}(\varepsilon/2)L(az) \le \frac{1}{|z-a|}(\varepsilon/2)2|z-a| = \varepsilon,$$
thanks to the triangle inequality. So for any continuous f on a domain $\mathcal{U}$, we see that f has a primitive on $\mathcal{U}$ if and only if $\oint_\gamma f = 0$ for any closed γ in $\mathcal{U}$ of type P_1, P_2, P_3, P_4. Thus the type P we were looking for are exactly the P_i's listed previously.

Corollary 14.1.

Now suppose f is a holomorphic function on an open set $\mathcal{U}$. Then for any near $z_0 \in \mathcal{U}$, we have $f(z) \sim f(z_0) + f'(z_0)(z-z_0) =: p(z)$. And any polynomial has a primitive on $\mathbb{C}$. So, it is natural to wonder if $\oint_\gamma f = 0$ for closed γ in $\mathcal{U}$ near z_0. First, we know that f being holomorphic on an open set $\mathcal{U}$ does *not* imply $\oint_\gamma f = 0$ for any closed γ in $\mathcal{U}$ – consider $f(z) = 1/z$. Indeed, we know
$$\oint_{C_1(0)} \frac{dz}{z} = 2\pi i \ne 0.$$
So what if $\oint_\gamma f$ for closed γ whose "inside" lies entirely in $\mathcal{U}$? As for the quick answer to this question, we will state for now that the answer lies in Green's theorem and line integrals. Recall that
$$\int_\gamma f(z)\,dz = \int_\gamma f(u+iv)\,(dx+idy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy. \tag{3}$$
Suppose $\gamma : [a,b] \to \mathbb{C}$ satisfies $\gamma(t) = x(t) + iy(t)$. Then $\gamma'(t) = x'(t) + iy'(t)$. So
$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b (u(x,y)+iv(x,y))\cdot(x'+iy')\,dt \\ &= \int_a^b (u(x,y)x'(t) - v(x,y)y'(t)\,dt,\end{aligned}$$
from which the seemingly "fake" integral (3) follows.

15. February 25

We know that if f is continuous on an open set $\mathcal{U}$, then f has a primitive on $\mathcal{U}$ if and only if $\oint_\gamma f = 0$ for all closed γ in $\mathcal{U}$. Now let's assume that f is holomorphic on $\mathcal{U}$. Then

$f \approx h(x)$ for some polynomial $h(x)$ locally; $h(x)$ has a primitive. To find out if f has local primitives in $\mathcal{U}$, it suffices to show that $\oint_\gamma f = 0$ for all closed γ in $\mathcal{U}$ near $z_0 \in \mathcal{U}$. We have

$$\int_\gamma f(z)\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy),$$

so we can apply Green's theorem on the two line integrals in $\mathbb{R}^2$ (on the RHS).

Theorem 15.1 (Green's theorem)**.** *Let $P, Q : \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions on an open set $\mathcal{U}$. Let γ be a simple, closed, piecewise-smooth curve on $\mathcal{U}$ with positive orientation such that the inside region R of γ is in $\mathcal{U}$. Then*

$$\oint_\gamma (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Corollary 15.1. *Suppose that f is holomorphic on an open set $\mathcal{U}$ and f' continuous on $\mathcal{U}$. Then for all simple closed piecewise-smooth γ in $\mathcal{U}$ such that the inside region of γ is in $\mathcal{U}$,*

$$\oint_\gamma f\,dz = 0.$$

Proof. This follows from applying the Cauchy-Riemann equation. □

This answer is somewhat unsatisfying in many ways – note that there are too many conditions needing to be satisfied:

(1) f' continuous (but we will see later that this is actually not needed)
(2) γ is simple
(3) we still lack the precise definition of what we mean by "inside"

So we will return to examining a function locally. Suppose that f is holomorphic on an open set $\mathcal{U}$, and let $z_0 \in U$. Since f is holomorphic at z_0, clearly

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

Let $f(z_0) =: a$ and $f'(z_0) =: b$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{f(z) - a}{z - z_0} - b\right| > \varepsilon$$

for any $z \in D^*(z_0, \delta)$. Letting $a + b(z - z_0) =: p(z)$, we can re-write the above inequality:

$$|f(z) - p(z)| \leq \varepsilon |z - z_0|.$$

Therefore for any radius $r < \delta$, for any closed γ, piecewise-smooth curve in $D(z_0, r)$, we have

$$\begin{aligned}\left|\oint_\gamma f\right| &= \left|\oint_\gamma f - \oint_\gamma p\right| = \left|\oint_\gamma (f - p)\right| \\ &\leq \oint_\gamma |f - p|\,|dz| \leq \varepsilon \oint_\gamma \underbrace{|z - z_0|}_{<r}\,|dz| < \varepsilon r L(r).\end{aligned}$$

Observe that $|z - z_0| < r$ since γ is wholly in $D(z_0, r)$. Suppose that $\gamma + \gamma_1 + \gamma_2$.

$$\left|\oint_\gamma f\right| \leq \left|\oint_{\gamma_1} f\right| + \left|\oint_{\gamma_2} f\right|$$

$$\leq 2 \max \left| \oint_{\gamma_i} f \right| \leq 2 \left| \oint_{\gamma_1} f \right|,$$

where we may assume without loss of generality that $|\oint_{\gamma_1} f| \geq |\oint_{\gamma_2} f|$. Evidently, things can go wrong if we are not cautious with choosing appropriate γ_1 and γ_2, so this needs to be done in a controlled manner. So it's better to stick to the "nice" curves. Some examples of nice curves:

(1) Let $R = [a,b] \times [c,d] \subset \mathbb{R}^2 = \{x+iy : x \in [a,b], y \in [c,d]\} \subset \mathbb{C}$. This set is a compact subset in $\mathbb{C}$. Then write

$$\partial R := \overline{z_1 z_2} + \overline{z_2 z_3} + \overline{z_3 z_4} + \overline{z_4 z_1} = z_1 z_3 + z_1 z_3^-,$$

so the rectangle consists of piecewise-smooth rectangular curves. Thus $L(\partial R) = 2(b-a) + 2(d-c)$; define $l_R = \frac{1}{2} L(\partial R) = L(z_1 z_3) = (b-a) + (d-c)$.

Notice for any rectangle R with $z_0, z \in R$, $|z - z_0| \leq l_R$. So $R \subseteq D(z_0, l_R)$. Now for f holomorphic at z_0, for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\left| \oint_{\partial R} f \right| < \varepsilon l_R L(\partial R) = 2\varepsilon l_R^2$$

for any rectangle $R \ni z_0$ with $l_R < \delta$ (since then $R \subset D(z_0, l_R) \subset D(z_0, \delta)$).

Assume that f is holomorphic on open $\mathcal{U}$, and assume also that the rectangle $R \subseteq \mathcal{U}$. Let $R = [a,b] \times [c,d]$, and $l = l_R$. Quarter R in the following way: Thus

$$\oint_{\partial R} f = \oint_{\partial R_{(1)}} f + \oint_{\partial R_{(2)}} f + \oint_{\partial R_{(3)}} f + \oint_{\partial R_{(4)}} f$$

and so

$$\left| \oint_{\partial R} f \right| \leq \sum_{j=1}^{4} \left| \oint_{\partial R_{(j)}} f \right| \leq 4 \max_{1 \leq j \leq 4} \left| \oint_{\partial R_{(j)}} f \right|.$$

Without loss of generality, suppose that $|\oint_{\partial R_{(1)}} f|$ is the biggest. If $l = l_R$, then $l_{R_1} = l/2$. Now R_1 can be quartered again; then one of its quarter rectangles R_2 is such that

$$\left| \oint_{\partial R_1} f \right| \leq 4 \left| \oint_{\partial R_2} f \right|.$$

So

$$\left| \oint_{\partial R} f \right| \leq 4 \left| \oint_{\partial R_1} f \right| \leq 4^2 \left| \oint_{\partial R_2} f \right|,$$

while $l_{R_2} = l_{R_1}/2 = l/2^2$. Continuing this operation, we obtain rectangles $R \supset R_1 \supset R_2 \supset \cdots$ such that

$$\left| \oint_{\partial R} f \right| \leq 4^n \left| \oint_{\partial R_n} f \right|,$$

and $l_{R_n} = l/2^n$ where $l = l_R$.

Theorem 16.1 (Goursat's theorem)**.** *Suppose that f is holomorphic on an open set $\mathcal{U}$; let $R \subset U$ be a rectangle. Then $\oint_{\partial R} f = 0$.*

Proof. Repeat the quartering process of R as we did previously. Then $R \supset R_1 \supset R_2 \supset \cdots$, so we have

$$\left|\oint_{\partial R} f\right| \le 4^n \left|\oint_{\partial R_n} f\right|,$$

and $l_{R_n} = l/2^n$ where $l := l_R$. Since each R_n is compact, we have

$$\bigcap_{n\ge 1} R_n \neq \emptyset.$$

So we can pick $z_0 \in \bigcap R_n$. Thus $z_0 \in \mathcal{U}$, and for each n we have $z_0 \in R_n$. Hence $R_n \subset D(z_0, l_{R_n})$. So for any $\varepsilon > 0$, since f is holomorphic at z_0, we know there is some $\delta > 0$ such that

$$\left|\oint_{\partial S} f\right| < 2\varepsilon l_S^2.$$

Hence for any rectangle $S \ni z_0$ with $l_s < \delta$, we can take n such that $2^n > l/\delta$. Then we have $l_{R_n} = l/2^n < \delta$, so

$$\left|\oint_{\partial R_n} f\right| < 2\varepsilon l_{R_n}^2 = 2\varepsilon(l^2/4^n).$$

From this it follows

$$\left|\oint_{\partial R} f\right| \le 4^n \left|\oint_{\partial R_n} f\right| < 2\varepsilon l^2.$$

This is indeed true for $\varepsilon > 0$, so the claim follows. □

Can we prove a Goursat-like theorem with weaker hypothesis? Suppose that f is holomorphic on $\mathcal{U} \setminus \{p\}$. If $R \subset \mathcal{U}$ and $p \in R^o$, does Goursat's theorem hold? The answer is no, since

$$\oint_{C_1(0)} \frac{dz}{z} \neq 0.$$

However, we may be able to salvage this by adding some extra conditions. Try to divide the rectangles as below. Then

$$\oint_{\partial R} f = \sum_{j=1}^{8} \oint_{\partial R_j} f + \oint_{\partial R_\delta} f = \oint_{\partial R_\delta} f,$$

for any $\delta > 0$, as

$$\oint_{\partial R_j} f = 0$$

for all $1 \le j \le 8$. So it suffices to estimate $\oint_{\partial R_\delta} f$. We can make an estimate of this integral by using the arc length of R_δ. So if we have $|f| \le C$ for some constant C near p, then for all sufficiently small δ ew have

$$\left|\oint_{\partial R} f\right| = \left|\oint_{\partial R_\delta} f\right| \le CL(R_\delta) = 8C\delta.$$

Therefore all we need is for f to be bounded on some $D^*(p,r)$. This is certainly true, for example, if f is continuous at p then $\lim_{z\to p} f(z)$ exists. But we can actually prove this with a weaker condition, namely

$$\lim_{z\to p}(z-p)f(z)=0.$$

For all $\varepsilon>0$, there is $\delta>0$ such that $R_\delta\subset R^o$, and $|z-p|\cdot|f(z)|<\varepsilon$ as long as $|z-p|<\delta$. Then for all $z\in\partial R_\delta$, we have

$$\frac{\delta}{2}\le|z-p|<\delta,$$

so

$$|f(z)|<\frac{\varepsilon}{|z-p|}\le\frac{\varepsilon}{\delta/2}$$

since $z\in D^*(p,\delta)$. Thus

$$\left|\oint_{\partial R}f\right|=\left|\oint_{\partial R_\delta}f\right|\le\frac{\varepsilon}{\delta/2}L(\partial R_\delta)=8\varepsilon.$$

Observe that this is true for any $\varepsilon>0$, so $\oint_{\partial R}f=0$. In conclusion, we proved Goursat's theorem with slightly weaker conditions.

Theorem 16.2 (Goursat's theorem, tweaked version I)**.** *Suppose that f is holomorphic on $\mathcal{U}\setminus\{p\}$ where $\mathcal{U}$ is open and $p\in\mathcal{U}$, and that*

$$\lim_{z\to p}(z-p)f(z)=0.$$

Then $\oint_{\partial R}f=0$ for any rectangle $r\subset\mathcal{U}$ with $p\in R^o$.

Corollary 16.1. *Suppose $p_1,p_2,\dots,p_n\in\mathcal{U}$ an open set, and that f is holomorphic on $\mathcal{U}\setminus\{p_1,\dot{;}sp_n\}$. Then the analogous result holds.*

We can even have the "bad" point *on* ∂R if we ensure that f is continuous at p. Again, for any $\delta>0$ we have

$$\oint_{\partial R}f=\oint_{\partial R_\delta}f.$$

If f is continuous on ∂R, then f is bounded near p; so suppose that $|f|\le M$. Hence

$$\left|\oint_{\partial R_\delta}f\right|\le ML(R_\delta)=4M\delta,$$

so $\oint_{\partial R}f=0$ as we wanted. In conclusion, we proved the following theorem.

Theorem 16.3 (Goursat's theorem, tweaked version II)**.** *Suppose that $\mathcal{U}$ is open, and that $p_1,\dots,p_k\in\mathcal{U}$. If f is holomorphic on $\mathcal{U}\setminus\{p_1,\dots,p_k\}$ and is continuous on $\mathcal{U}$, then*

$$\oint_{\partial R}f=0$$

for any rectangle $R\subset U$.

Recall that we hav suspected whether the following claim is true or not: if f is holomorphic on an open set $\mathcal{U}$, then $\oint_\gamma f=0$ for all closed γ in $\mathcal{U}$ which is wholly inside of $\mathcal{U}$. So far, we proved this for any rectangles in $\mathcal{U}$. But is it possible to construct local primitives for f using rectangles?

17. March 1

Suppose f is holomorphic on $D = D(z_0, R)$ (except possibly at finitely many points where f is continuous). Then we can use corner curves to build a primitive F on D: for $z \in D$ define

$$F(z) = \int_{z_0 z} f.$$

Note that this is well-defined (i.e., unique for every corner curve), since

$$F(z) = \int_{z_0 z} f = \int_{z_0 a} f + \underbrace{\oint_{\partial R} f}_{=0\,(\text{by Goursat})} + \int_{az} f = F(a) + \int_{az} f.$$

Therefore,

$$\frac{F(z) - F(a)}{z - a} = \frac{1}{z - a} \int_{az} f.$$

So by the same argument as before (i.e., FTC I), we have $F'(a) = f(a)$.

Theorem 17.1 (Cauchy's theorem for a disc)**.** *Suppose f is a holomorphic on disc D except possibly at finitely many points, where f is continuous. Then f has a primitive on D. Thus,*

$$\oint_\gamma f = 0$$

for all closed curve γ in D.

17.1. Integrals along continuous curves for holomorphic functions

Suppose that f is holomorphic on an open set $\mathcal{U}$ (except possibly at finitely many points, where f is continuous). Then by Cauchy's theorem for disc, f has a primitive on each disc in $\mathcal{U}$. Thus $\int_\gamma f$ is well-defined for all continuous curves γ in $\mathcal{U}$. So we have the natural properties we expect, such as

(1) $\int_\gamma (f + g) = \int_\gamma f + \int_\gamma g$
(2) $\int_\gamma \lambda f = \lambda \int_\gamma f$ for all $\lambda \in \mathbb{C}$
(3) $\int_{\gamma^-} f = -\int_\gamma f$
(4) If $\gamma_1, \dots, \gamma_n$ are joinable curves in $\mathcal{U}$, then

$$\int_{\gamma_1 + \cdots + \gamma_n} f = \sum_{j=1}^n \int_{\gamma_j} f.$$

(5) If $\gamma \equiv z_0$ is a constant curve, then $\int_\gamma f = 0$.

However, note that it does *not* necessarily follow that for $|f| \le M$,

$$\left| \int_\gamma f \right| \le M L(\gamma).$$

Definition 17.1. If $\displaystyle\sum_{j=1}^n |z_i - z_j|$ is bounded, then for any partitions $a = z_0 < \cdots < z_n = b$, then we can define $L(\gamma)$. Such curve is said to be a *rectifiable curve*.

Also, we can use the change of variables (say g). So if γ is a curve in $\mathcal{V}$, then

$$\int_{g\circ\gamma} f = \int_{\gamma} f(g(w))g'(w)\,dw.$$

In particular, if f is holomoaphic on $\mathcal{U}$ open with $w \in \mathbb{C}$, if $g(z) = z - w$, then

$$\int_{\gamma} f(z)\,dz = \int_{\gamma+w} f(z-w)\,dw.$$

17.2. **Homotopic paths**

Suppose $\mathcal{U}$ is open, and $\gamma_0, \gamma_1 : [a,b] \to \mathcal{U}$ are continuous.

Definition 17.2. We say γ_0 and γ_1 are *homotopic in* $\mathcal{U}$ if there exists a continuous map $\varphi : [0,1] \times [a,b] \to \mathcal{U}$ such that

$$\varphi(0,t) = \gamma_0(t) \forall t \in [a,b]$$
$$\varphi(1,t) = \gamma_1(t) \forall t \in [a,b].$$

Note that for each $s \in [0,1]$, we have $\gamma_s(t) = \varphi(s,t)$, which is continuous on $\gamma_s : [a,b] \to \mathcal{U}$. Thus the family $\{\gamma_s\}_{s\in[0,1]}$ is a continuous deformation of γ_0 to γ_1.

Definition 17.3. The map φ above is called a *homotopy* in $\mathcal{U}$ of γ_0 and γ_1.

Definition 17.4. γ_0 and γ_1 are *homotopic in* $\mathcal{U}$ *with fixed endpoints* if there exist $p, q \in \mathcal{U}$ such that $\gamma_s(a) = p$ and $\gamma_s(b) = q$ for all $s \in [0,1]$. If this is the case, then we write $\gamma_0 \sim \gamma_1$ in $\mathcal{U}$.

Remark. One can verify that $\sim$ is an equivalence relation on all paths in $\mathcal{U}$ from p to q, which will be left as an exercise.

When we say that paths with the same endpoints are *homotopic*, we will always mean in the following sense (except the case when they are closed curves).

Definition 17.5. γ_0 and γ_1 are *homotopic in* $\mathcal{U}$ *as closed paths* if each γ_s is closed in the homotopy φ. We write $\gamma_0 \sim \gamma_1$ in $\mathcal{U}$.

Definition 17.6. A closed path γ_0 is *homotopic to a point in* $\mathcal{U}$ if there exists a constant path $\gamma_1 \equiv p \in \mathcal{U}$ such that $\gamma_0 \sim \gamma_1$ in $\mathcal{U}$ (as closed paths).

Definition 17.7. A domain $\mathcal{U}$ is *simply connected* if every closed path in $\mathcal{U}$ is homotopic to a point in $\mathcal{U}$. To put it intuitively, $\mathcal{U}$ has no "holes".

Example. $\mathbb{C} \setminus \{0\}$ is not simply connected.

Proposition 17.1. *A domain* $\mathcal{U}$ *is simply connected if and only if any two paths with the same endpoints in* $\mathcal{U}$ *are homotopic in* $\mathcal{U}$.

Example. Recall that a set E is convex if $\overline{zw} \in E$ for any $z, w \in E$. Any convex open set is simply connected.

Definition 17.8. A set $E$is said to be *star-shaped* if there exists $p \in E$ such that $\overline{pz} \subset E$ for all $z \in E$.

Example. Any star-shaped open sets are simply connected. One example of a star-shaped set is $\mathbb{C} \setminus \mathrm{Ray}_\theta$.

Example. If γ is a path in an open set $\mathcal{U}$, one can show that $\gamma \sim \tilde{\gamma}$ in $\mathcal{U}$ (as fixed endpoint, or as closed paths) for some piecewise-smooth $\tilde{\gamma}$ in $\mathcal{U}$.

Definition 17.9. Given f on an open set $\mathcal{U}$, a *primitive for f along a homotopy* is a function $p : [c,d] \times [a,b] \to \mathbb{C}$ such that for each $(s,t) \in [c,d] \times [a,b]$ there is an open set $(s,t) \in I$, a disc $\varphi(s,t) \in D$, and a primitive F for f on D such that $P = F \circ \varphi$ on I.

Proposition 17.2. *Suppose that p is a primitive for f along a homotopy φ on an open set $\mathcal{U}$.*

(1) p is continuous
(2) q is another primitive for f along φ if and only if $p - q$ is a constant function.
(3) For all $s \in [c,d]$, the function $p_s : [a,b] \to \mathbb{C}$ defined by $p_s(t) := p(s,t)$ is a primitive for f along the path $\gamma_s(t) = \varphi(s,t)$.

18. March 4

18.1. Building a primitive along a homotopy

Suppose that $\mathcal{U}$ is open, and that f has a primitive on each disc in $\mathcal{U}$. Can we build a primitive for f along any homotopy $\varphi : [c,d] \times [a,b] \to \mathcal{U}$?

We want to partition the rectangles into multiple smaller rectangles, and map to $\mathcal{U}$ via homotopy φ; find an open disc for each of the mapped small rectangles (see the diagram below).

Proposition 18.1. *There exist discs $D_{k,j}$ following partitions $c = s_0 < s_1 < \cdots < s_m = d$ and $a = t_0 < t_1 < \cdots < t_n = b$ along φ in $\mathcal{U}$. In other words, for all $1 \le k \le m$ and $1 \le j \le n$, we have $\varphi([s_{k-1}, s_k] \times [t_{j-1}, t_j] \subseteq D_{k,j}$.*

Let $R = [c,d] \times [a,b]$. Since R is compact and φ continuous, $\varphi(R)$ is compact as well. So $\varphi(R) \subset \mathcal{U}$ is a compact subset of an open set, so there is a minimal distance $\varepsilon > 0$ from the points of $\varphi(R)$ to $\mathcal{U}^c$. Hence, φ is continuous on a compact set R, so φ is uniformly continuous on R. In conclusion, there is $\delta > 0$ such that $\varphi(D((s,t),\delta)) \subset D(\varphi(s,t),\varepsilon)$ for each $(s,t) \in R$. For example, one can take n,m large enough so that $(b-a)/n, (d-c)/n < \delta/\sqrt{2}$, and use $D_{kj} = D(\varphi(s_k,t_j),\varepsilon)$.

Now suppose D_{kj} follow

$$c = s_0 < s_1 < \cdots < s_m = d$$
$$a = t_0 < t_1 < \cdots < t_n = b.$$

along a homotopy $\varphi : [c,d] \times [a,b] \to \mathcal{U}$. If F has a primitive on each disc in $\mathcal{U}$, then for each $k = 1, \ldots, m$, one can map each rectangle in the following manner, sequentially:

Proposition 18.2. *There are primitives $F_1, \ldots, F_n$ for f on discs $D_{k,1}, D_{k,2}, \ldots, D_{k,n}$ respectively such that $F_{j-1} = F_j$ on $D_{j-1} \cap D_j$ for $j = 2, \ldots, n$. Therefore, we can define $p_k : [s_{k-1}, s_k] \times [a,b] \to \mathbb{C}$ by setting $p_k = F_j \circ \varphi$ on R_j. The p_k defines a primitive for f along $\varphi : [s_{k-1}, s_k] \times [a,b] \to \mathcal{U}$.*

Now we have a primitive p_k for f along each $\varphi : [s_{k-1}, s_k] \times [a,b] \to \mathcal{U}$. After constructing for each rectangle, we can adjust each of the primitives by a constant so that they match on the edges $s = s_k$. So let $q_1 = p_1$. Notice that both $q_1(s_1, \cdot)$ and $p_2(s_1, \cdot)$ are both primitives for f along the path $\gamma_s(t) := \varphi(s,t)$. Therefore for some C, $q_1(s_1, \cdot) = p_2(s_1, \cdot) + C$ on

$[a,b]$. So write $q_2 = p_2 + C$. Then q_2 is a primitive for f along $\varphi : [s_1, s_2] \times [a,b] \to \mathcal{U}$, and $q_2 = q_1$ when $s = s_1$. We may continue in this way to obtain primitives q_k for f along $\varphi : [s_{k-1}, s_k] \times [a,b] \to \mathcal{U}$ so that $q_{k-1} = q_k$ when $s = s_{k-1}$.

Define $p : [c,d] \times [a,b] \to \mathbb{C}$ by setting $p = q_k$ on $[s_{k-1}, s_k] \times [a,b]$.

Proposition 18.3. *p is a primitive for f along $\varphi : [c,d] \times [a,b] \to \mathcal{U}$.*

In conclusion, we proved that

Theorem 18.1. *Let $\mathcal{U}$ be an open set, and f has a primitive on each disc in $\mathcal{U}$. Also suppose that $\varphi : [0,1] \times [a,b] \to \mathcal{U}$ is a homotopy in $\mathcal{U}$. Then there exists a primitive p for f along φ.*

Remark. If φ is a fixed endpoint homotopy, i.e., $\varphi_s(a) = \varphi(s,a) = z_1$ and $\varphi_s(b) = \varphi(s,b) = z_2$ for all $s \in [0,1]$, then p must be constant on top and the bottom edges.

19. March 6 & 8

Last time, for any $\mathcal{U}$ open and $\varphi : [0,1] \times [a,b] \to \mathcal{U}$ homotopy, with f having a primitive for every disc in $\mathcal{U}$, we proved taht there exists a primitive $p : [0,1] \times [a,b] \to \mathbb{C}$ for f along φ.

Recall that for all $s \in [0,1]$, if $\gamma_s : \varphi(s,\cdot) : [a,b] \to \mathcal{U}$ is a path, then we defined $p_s := p(s,\cdot) : [a,b] \to \mathbb{C}$ to be a primitive for f along γ_s. If φ is a fixed point homotopy (i.e., $\gamma_s(a) = z_1$ and $\gamma_s(b) = z_2$, then for all s, both $p_s(a)$ and $p_s(b)$ are constant. Thus $p_s(b) - p_s(a)$ has the same value across all s.

Proposition 19.1. *If φ is a closed pat homotopy such that $\gamma_s(a) = \gamma_s(b)$ for all s, then we also have $p_s(b) - p_s(a) \equiv C$ for some constant C for all s.*

19.1. Homotopy up to reparametrization

Suppose $\mathcal{U}$ is open, and γ_0 and γ_1 are curves in $\mathcal{U}$.

Proposition 19.2. *If there are parametrizations $\gamma_0, \gamma_1 : [a,b] \to \mathcal{U}$ such that $\gamma_0 \sim \gamma_1$ (with fixed endpoints, or as closed paths) then for any interval $[c,d]$ and reparametrizations $\widetilde{\gamma}_0, \widetilde{\gamma}_1 : [c,d] \to \mathcal{U}$ of γ_0 and γ_1 respectively, we have $\widetilde{\gamma}_0 \sim \widetilde{\gamma}_1$ (with fixed endpoints or as closed paths reparametrization).*

Definition 19.1. Suppose that γ_0 and γ_1 satisfy the property as outlined in the above proposition. Then we say that the curves γ_0 and γ_1 are *homotopic in $\mathcal{U}$* (with fixed endpoints or as closed paths).

Now we are ready to prove Cauchy's theorem.

Theorem 19.1 (Cauchy's theorem)**.** *Let f be a holomorphic function on an open set $\mathcal{U}$ (or holomorphic on $\mathcal{U}$ except for finitely many points with f being continuous on $\mathcal{U}$ – call this property (†)). Let γ_0 and γ_1 be curves in $\mathcal{U}$.*

(i) If $\gamma_0 \sim \gamma_1$ in $\mathcal{U}$ with fixed endpoints, or

(ii) If $\gamma_0 \sim \gamma_1$ in $\mathcal{U}$ as closed curves,

then

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

In particular, (iii) if $\gamma \sim p$ where $p \in \mathcal{U}$, then

$$\int_\gamma f = 0.$$

Proof. (iii) clearly follows from (ii), so it suffices to prove that the claim holds for (i) and (ii). Let $\gamma_0 \sim \gamma_1$ in $\mathcal{U}$ with fixed endpoints or as closed curves. Then for given parametrizations $\gamma_0, \gamma_1 : [a,b] \to \mathcal{U}$, we have a homotopy (fixed endpoints or closed curve) $\varphi : [0,1] \times [a,b] \to \mathcal{U}$. Since f is holomorphic in $\mathcal{U}$ or has the (†) property, f has primitives on each disc in $\mathcal{U}$. Thus there exists a primitive for f along φ; call this primitive $p : [0,1] \times [a,b] \to \mathbb{C}$. And then p_0 and p_1 are primitives for f along φ_0 and φ_1 respectively, and also $p_0(b) - p_0(a) = p_1(b) - p_1(a)$. Therefore

$$\int_{\gamma_0} f = p_0(b) - p_0(a) = p_1(b) - p_1(a) = \int_{\gamma_1} f. \qquad \square$$

Definition 19.2. A set $\mathcal{U}$ is a *simply connected domain* if $\mathcal{U}$ is a domain such that for any closed curve γ, there exists a point $p \in \mathcal{U}$ such that $\gamma \sim p$. Intuitively, a simply connected domain has no "holes".

Corollary 19.1 (Cauchy's theorem for simply connected domains)**.** *Let f be holomorphic or has the (†) property on $\mathcal{U}$, where $\mathcal{U}$ is a simply connected domain. Then for all closed curve $\gamma \in \mathcal{U}$, we have*

$$\oint_\gamma f = 0.$$

Therefore, f has a primitive in $\mathcal{U}$.

Example (Nudging a curve). Suppose f has the (†) property or is holomorphic on an open set $\mathcal{U}$ and γ closed curve in $\mathcal{U}$ with distance $\varepsilon > 0$ to $\mathcal{U}^c$. Suppose $|w| < \varepsilon$. Then $\gamma \sim \gamma + w$ in $\mathcal{U}$ as closed curves. Therefore,

$$\int_\gamma f = \int_{\gamma+w} f$$

for all $|W| < \operatorname{dist}(\gamma, \mathcal{U}^c)$.

Example (Nudging a function). Let f be holomorphic on an open set $\mathcal{U}$ or has the (†) property. Let γ be a closed curve in $\mathcal{U}$. Then for all $|w| < \operatorname{dist}(\gamma, \mathcal{U}^c)$, we have

$$\int_\gamma f(z)\,dz = \gamma_{\gamma-w} f(z)\,dz$$

by nudging a curve. Then by the change of variables,

$$\int_{\gamma-w} f(z)\,dz = \int_\gamma f(z-w)\,dz.$$

Therefore,

$$\int_\gamma f(z)\,dz = \int_\gamma f(z-w)\,dz$$

for all $|w| < \operatorname{dist}(\gamma, \mathcal{U}^c)$.

Example (Change in log or arg). For any curve γ curve in $\mathbb{C} \setminus \{0\}$, we have $\delta \log \gamma = l(b) - l(a) = \int_\gamma \frac{dz}{z}$, since l is a primitive for z^{-1} along γ. As for argument, $\delta \arg \gamma = \alpha(b) - \alpha(a)$ where l and α are branches of log and arg respectively along a parametrization $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ of γ. Recall that $\delta \arg \gamma$ and $\delta \log \gamma$ are independent of parametrization of γ or branches.

Suppose that γ is closed. Then $\delta \arg \gamma = \alpha(b) - \alpha(a) = 2\pi k$ for some $kin\mathbb{Z}$, since $\alpha(a), \alpha(b) \in \arg z_0$. k is equal to the number of times the curve γ wraps around 0. Also, $l(a)$ and $l(b)$ have the same real part $\ln |z_0|$, so

$$\int_\gamma \frac{dz}{z} = \delta \log \gamma = l(b) - l(a) = i(\alpha(b) - \alpha(a)) = i\delta \arg \gamma = i(2\pi k).$$

The example above prompts us to introduce winding numbers.

Definition 19.3. Suppose that γ is a closed curve in $\mathbb{C}$, and that $z_0 \in \mathbb{C} \setminus \gamma$. The *winding number* (or *index*) of γ around z_0 is

$$I(\gamma, z_0) := \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0}.$$

Now we shall give a more rigorous and precise definition of the "inside" and "outside" of a closed curve using winding numbers.

Definition 19.4. Let γ be a closed curve. The *inside* of γ is $\{z \in \mathbb{C} \setminus \gamma : I(\gamma, z) \neq 0\}$; the *outside* of γ is $\{z \in \mathbb{C} \setminus \gamma : I(\gamma, z) = 0\}$.

Proposition 19.3. *Let $E \subseteq \mathbb{C} \setminus \gamma$. If E is connected and unbounded, then E is in the outside of γ. However, the converse is not true.*

While the proof of the theorem below is beyond the scope of this course, we nonetheless present it as it is an important result regarding simple curves.

Theorem 19.2 (Jordan curve theorem)**.** *Let γ be a simple closed curve. Then γ divides the plane into two connected components: first, the inside, which is bounded and simply connected with $I(\gamma, z) = \pm 1$ for any z inside γ; and second, the outside, which is unbounded with $I(\gamma, z) = 0$ for z outside γ. Furthermore, γ is the common boundary of these two regions.*

This theorem may look "simple", but one should note that a simple closed curve may not be as "simple" as one thinks – for instance, the diagram below is an example of a simple closed curve.

Proposition 19.4 (Properties of a winding number)**.** *Let γ be a closed curve, and $z_0 \in \mathbb{C} \setminus \gamma$.*

(i) $I(\gamma, z_0) \in \mathbb{Z}$

(ii) Let $\mathcal{U} = \mathbb{C} \setminus \{z_0\}$. For any $|h| < \operatorname{dist}(z_0, \gamma) = \varepsilon$,

$$I(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - h - z_0} = I(\gamma, z_0 + h).$$

In other words, $I(\gamma, \cdot)$ is locally constant on $\mathbb{C} \setminus \gamma$.

(iii) Therefore, $I(\gamma, \cdot)$ is constant on each connected component of $\mathbb{C} \setminus \gamma$.

Proof. By the change of variables,

$$\frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-z_0} = \frac{1}{2\pi i}\oint_{\gamma-z_0}\frac{dz}{z} = \frac{1}{2\pi i}\delta\log(\gamma - z_0).$$

The claim follows upon noting that $\delta\log\gamma$ is of the form $2\pi k i$ for some $k\in\mathbb{Z}$. $\square$

Proposition 19.5 (Winding number for homotopic curves)**.** *Let γ and η be closed curves such that $z_0\notin\gamma$ and $z_0\notin\eta$.*
(i) If $\gamma\sim\eta$ in $\mathbb{C}\setminus\{z_0\}$, then $I(\gamma,z_0)=I(\eta,z_0)$.
(ii) If $\gamma\sim p$ for some point $p\in\mathbb{C}\setminus\{z_0\}$, then $I(\gamma,z_0)=0$.

19.2. **Cauchy integration formula**

Suppose f is holomorphic on $\mathcal{U}$ open. Also, let $\gamma\sim p$ where $p\in\mathcal{U}$; let $a\in\mathcal{U}$ not in γ. The function

$$g(z) := \begin{cases} \frac{f(z)-f(a)}{z-a} & z\in\mathcal{U}\setminus\{a\} \\ f'(a) & z=a \end{cases}$$

is holomorphic on $\mathcal{U}\setminus\{a\}$, and is continuous on $\mathcal{U}$. Therefore, by Cauchy's theorem,

$$\oint_\gamma g = 0.$$

But then

$$\oint_\gamma g = \oint_\gamma \frac{f(z)-f(a)}{z-a}\,dz = \oint_\gamma \frac{f(z)}{z-a}\,dz - f(a)\oint_\gamma\frac{dz}{z-a}.$$

Therefore, we have

$$f(a)I(\gamma,a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\,dz.$$

So the values of f on γ completely determine the values of f inside γ.

We shall state this central result explicitly.

Theorem 19.3 (Cauchy integration formula)**.** *Suppose f is holomorphic on an open set $\mathcal{U}$ open, and $\gamma\sim p$ where $p\in\mathcal{U}$. Then for all $z\in\mathcal{U}$ not on γ,*

$$f(z)I(\gamma,z) = \frac{1}{2\pi i}\oint_\gamma\frac{f(w)}{w-z}\,dw.$$

Notice that

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\frac{1}{1-\frac{z-a}{w-a}} = \frac{1}{w-a}\sum_{n=0}^\infty\left(\frac{z-a}{w-a}\right)^n.$$

Thus as long as $|z-a|<|w-a|$, This holds if $z\in D(a,\mathrm{dist}(a,\gamma))$. So if we could switch the order of γ and σ, then we might obtain

$$\begin{aligned} f(z)I(\gamma,z) &= \frac{1}{2\pi i}\oint_\gamma\frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\oint_\gamma\sum_{n=0}^\infty\frac{f(w)}{w-a}\left(\frac{z-a}{w-a}\right)^n \end{aligned}$$

$$\stackrel{?}{=} \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-a)^{n+1}}\, dw \right] (z-a)^n,$$

which is a power series expansion for f about a. This would tell us that holomorphicity implies analyticity, and would also tell us a formula for higher derivatives. So the question is: when can we switch σ and $\oint$? For this we need to discuss a notion called *uniform convergence*.

19.3. **Uniform convergence**

Definition 19.5. $f : \mathbb{C} \to \mathbb{C}$ is *bounded* on a set E if there exists $C \geq 0$ such that $f(z) \leq C$ for all $z \in E$. The *uniform norm* (or *sup norm*) of f on E is

$$\|f\| = \|f\|_E = \sup_{z \in E} |f(z)|.$$

Definition 19.6. We say $\{f_n\}$ *converges to* f *uniformly on* E if

$$\lim_{n\to\infty} \|f_n - f\| = 0,$$

i.e., for all $\varepsilon > 0$ there exists $N > 0$ such that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and $z \in E$. So both of the conditions below must hold:

- $\lim f_n(z) = f(z)$ at each $z \in E$
- The convergence is uniform: given $\varepsilon > 0$ the same N works at every $z \in E$.

Definition 19.7. We say

$$\sum_{n=0}^{\infty} f_n = f$$

uniformly on E if

$$\lim_{n\to\infty} \sum_{j=0}^{n} f_j = f$$

uniformly on E.

20. March 13

Suppose $\{f_n\}$ is a sequence of functions continuous on γ a piecewise-smooth curve. If $f_n \to f$ uniformly on γ then

$$\begin{aligned}
\left| \int_\gamma f_n - \int_\gamma f \right| &= \left| \int_\gamma (f_n - f) \right| \\
&\leq \int_\gamma |f_n - f|\,|dz| \\
&\leq \|f_n - f\|_\gamma L(\gamma) \to 0.
\end{aligned}$$

So if $\sum f_n = f$ uniformly on γ,

$$\sum_{n=0}^{\infty} \int_\gamma f_n = \lim_{n\to\infty} \sum_{j=0}^{n} \int_\gamma f_j = \lim_{n\to\infty} \int_\gamma \sum_{j=0}^{n} f_j$$

$$= \int_\gamma \lim_{n\to\infty} \sum_{j=0}^{n} f_j \quad (\because \text{ uniform convergence})$$

$$= \int_\gamma \sum_{n=0}^{\infty} f_n.$$

20.1. **Uniform convergence of power series**

Let $\sum a_n z^n$ be a power series.

Lemma 20.1. *If the sequence $\{a_n z_0^n\}$ is bounded for some $z_0 \neq 0$ then $\sum a_n z^n$ converges absolutely, and uniformly on $\overline{D(0,r)}$ for all $0 < r < |z_0|$.*

Proof. Let $r_0 = |z_0|$. Then there is $C > 0$ such that $|a_n| r_0^n = |a_n z_0^n| \leq C$ for all n. Thus $|a_n| \leq C/r_0^n$ for all n. Let $0 < r < r_0$. Then for $z \in \overline{D(0,r)}$, we have

$$|a_n z^n| = |a_n||z|^n = (C/r_0^n) r^n = C\left(\frac{r}{r_0}\right)^n$$

for all n. For each n,

$$\|a_n z^n\|_{\overline{D(0,r)}} \leq C\left(\frac{r}{r_0}\right)^n.$$

Now by the comparison test,

$$\sum_{n=0}^{\infty} \|a_n z^n\|_{\overline{D(0,r)}} \leq C \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n.$$

Since $0 < r/r_0 < 1$, the geometric series $C\sum (r/r_0)^n$ converges. Therefore $\sum a_n z^n$ converges absolutely, and uniformly on $\overline{D(0,r)}$ as desired. $\square$

Corollary 20.1. *Suppose that the series $\sum a_n z^n$ has radius of convergence $R > 0$. Then $\sum a_n z^n$ converges uniformly for each $\overline{D(0,r)}$ for all $0 < r < R$. By translation, for any z_0 the same result holds for $\sum a_n (z - z_0)^n$.*

Theorem 20.1 (Cauchy-type integrals)**.** *Let γ be a piecewise-smooth curve, and g continuous on γ. Then*

$$G(z) = \int_\gamma \frac{g(w)}{w - z}\, dw$$

is analytic *on $\mathbb{C} \setminus \gamma$ and*

$$G^{(n)}(z) = n! \int_\gamma \frac{g(w)}{(w - z)^{n+1}}\, dw.$$

Moreover, the power series expansion at each $a \in \mathbb{C} \setminus \gamma$ is valid for all

$$|z - a| < \operatorname{dist}(a, \gamma).$$

Proof. Let $a \in \mathbb{C} \setminus \gamma$. Then let $\varepsilon = \operatorname{dist}(a, \gamma) > 0$. Let $z \in D(a, \varepsilon)$. Then

$$r = \frac{|z - a|}{\varepsilon} < 1,$$

since $|z - a| < r\varepsilon$ and $|w - a| \geq \varepsilon$. So for every $w \in \gamma$, we have

$$\left|\frac{z - a}{w - a}\right| < \frac{r\varepsilon}{\varepsilon} = r < 1.$$

Thus $h(\gamma) \subset \overline{D(0,r)}$ where $h(w) = (z-a)/(w-a)$. Since the geometric series

$$\frac{1}{1-\zeta} = \sum_{n=0}^{\infty} \zeta^n$$

converges uniformly on $\overline{D(0,r)}$, we have

$$\frac{w-a}{w-z} = \frac{1}{1-\frac{z-a}{w-a}} = \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n$$

uniformly for all $w \in \gamma$. And since $g(w)/(w-a)$ is continuous on γ and γ is compact, it follows that it is bounded on γ. Hence

$$\frac{g(w)}{w-z} = \frac{g(w)}{w-a} \cdot \frac{w-a}{w-z} = \sum_{n=0}^{\infty} \frac{g(w)}{w-a} \left(\frac{z-a}{w-a}\right)^n$$

uniformly for $w \in \gamma$. All in all,

$$G(z) = \int_{\gamma} \frac{g(w)}{w-z}\, dw = \sum_{n=0}^{\infty} \left(\int_{\gamma} \frac{g(w)}{(w-a)^{n+1}}\, dw \right) (z-a)^n.$$

Therefore $\sum a_n (z-a)^n$ converges with coefficients

$$a_n = \int_{\gamma} \frac{g(w)}{(w-a)^{n+1}}\, dw$$

converges to $G(z)$ on $D(a,\varepsilon)$. Thus $G^{(n)}(a) = n! a_n$. □

21. March 15

We start with a corollary to the theorem on Cauchy-type integrals and the Cauchy integration formula.

Corollary 21.1 (Taylor's theorem)**.** *Let f be holomorphic on an open set $\mathcal{U}$. Then f is analytic on $\mathcal{U}$, and for all $a \in \mathcal{U}$, the power series expansion for f at a is valid on $D(a, \operatorname{dist}(a, \mathcal{U}^c))$.*

Proof. Let $a \in \mathcal{U}$, and let $\varepsilon := \operatorname{dist}(a, \mathcal{U}^c) > 0$. Then for $0 < r < \varepsilon$, let $\gamma = C_r(a)$. Then $\gamma \sim p$ for some $p \in \mathcal{U}$ since $D(a,\varepsilon) \subset \mathcal{U}$. So by the Cauchy integration formula, we have

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-z)}\, dw$$

since $I(\gamma, z) = 1$. But f is continuous on γ, and γ is smooth, so

$$G(z) = \oint_{\gamma} \frac{f(w)}{w-z}\, dw$$

has a power series expansion about a on $|z-a| < \operatorname{dist}(a,\gamma)$. Hence f has a power series expansion on $D(a,r)$ for each $0 < r < \varepsilon$, so the same claim holds for $D(a,\varepsilon)$ as well. □

Corollary 21.2 (Cauchy integration formula for higher derivatives)**.** *Let f be holomorphic on an open set $\mathcal{U}$, and let $\gamma \sim p$ for some point $p \in \mathcal{U}$. Then fro each $z \in \mathcal{U} \setminus \gamma$, we have*

$$2\pi i \frac{f^{(n)}(z)}{n!} I(\gamma, z) = \oint_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw$$

for $n = 0, 1, 2, \ldots$.

Proof. The $n = 0$ case is covered by the Cauchy integration formula, so it suffices to prove the claim for $n \geq 1$. First, we prove the claim for piecewise-smooth γ; after that, use homotopy to get for γ continuous (exercise in an assignment). □

Now we are ready to prove the converses of some big theorem covered previously. At this point, we know that holomorphicity implies analyticity, which in turn implies that the given function is infinitely differentiable. Taylor's theorem proves analyticity implies holomorphicity, so we know holomorphicity and analyticity are equivalent. Also, if f has a primitive F on $\mathcal{U}$, then F is holomorphic on $\mathcal{U}$, so F is infinitely differentiable. Hence $f = F'$ is differentiable hence holomorphic. Thus, the following facts give Morera's theorem. There are multiple ways this theorem can be stated; we present four different ways it can be stated.

Theorem 21.1 (Morera's theorem)**.** *Let f be continuous on an open set $\mathcal{U}$.*

(i) (Converse of Goursat's theorem) If $\oint_{\partial R} f = 0$ for all rectangles $R \subset \mathcal{U}$, then f has a primitive on each disk in $\mathcal{U}$. Therefore f is holomorphic in $\mathcal{U}$.

(ii) If f has local primitives in $\mathcal{U}$, then f is holomorphic on $\mathcal{U}$.

(iii) If $\oint_\gamma f = 0$ for all closed curve $\gamma \in \mathcal{U}$, then f has a primitive in $\mathcal{U}$. Therefore f is holomorphic on $\mathcal{U}$.

(iv) (Converse of Cauchy's theorem) If $\oint_\gamma f = 0$ for all $\gamma \sim p$ in $\mathcal{U}$ for some $p \in \mathcal{U}$, then $\oint_{\partial R} f = 0$ for all $R \subset \mathcal{U}$. Therefore f is holomorphic on $\mathcal{U}$.

Corollary 21.3. *Let f be continuous on an open set $\mathcal{U}$. If f is holomorphic on $\mathcal{U}$ except for finitely many points, then f is holomorphic on $\mathcal{U}$.*

Proof. $\oint_{\partial R} f = 0$ for all rectangles $R \subset \mathcal{U}$ by tweaked Goursat's theorem, so f is holomorphic on $\mathcal{U}$ by Morera's theorem. □

We present more corollaries of the Cauchy integration formula.

Corollary 21.4 (Cauchy's inequalities)**.** *Let f be holomorphic on an open set $\mathcal{U}$, and $D(z_0, r) \subseteq \mathcal{U}$. Then*

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \|f\|_{C_r(z_0)}.$$

Proof. Since $\gamma = C_r(z_0) \sim p$ for some $p \in \mathcal{U}$, we may apply the Cauchy integration formula at centre z_0:

$$\begin{aligned}\left|f^{(n)}(z_0) I(\gamma, z_0)\right| = \left|f^{(n)}(z_0)\right| &= \frac{n!}{2\pi} \left| \oint_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\, dw \right| \\ &\leq \frac{n!}{2\pi} \oint_\gamma \frac{|f(w)|}{|w-z_0|^{n+1}}\, |dw|\end{aligned}$$

$$\leq \frac{n1}{2\pi}\frac{\|f\|_{C_r(z_0)}}{r^{n+1}}L(\gamma) \quad (\because w \in \gamma = C_r(z_0))$$
$$= \frac{n!}{2\pi}\frac{\|f\|_{C_r(z_0)}}{r^{n+1}}(2\pi r) = n!\frac{\|f\|_{C_r(z_0)}}{r^n}. \qquad \square$$

Theorem 21.2 (Liouville's theorem)**.** *Every bounded entire function is constant.*

Proof. Suppose f is entire, and that there exists a constant C so that $|f| \leq C$. Suppose $z_0 \in \mathbb{C}$. Since f is entire, we can apply Cauchy's inequality on any disk $\overline{D(z_0, r)}$. For $n = 1$,

$$|f'(z_0)| \leq \frac{\|f\|_{C_r(z_0)}}{r} \leq \frac{C}{r}.$$

Since this is true for any $r > 0$, it follow $|f'(z_0)| = 0$. Hence $f' \equiv 0$ on $\mathbb{C}$, so f must be constant on $\mathbb{C}$. □

Theorem 21.3 (Fundamental theorem of algebra)**.** *Every non-constant polynomial* $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ *for* $a_j \in \mathbb{C}$ *and* $a_n \neq 0$ *has a root in* $\mathbb{C}$.

Proof. Suppose otherwise. Then $p \neq 0$ on $\mathbb{C}$, so $1/p(z)$ must be entire. Then $1/p(z)$ is bounded (think about why). So by Liouville, $1/p(z)$ is constant; so $p(z)$ must be a constant, which is a contradiction. □

22. March 18 & 20

Thanks to the fundamental theorem of algebra, every polynomial of degree $k \geq 1$ is of the form

$$p(z) = a(z - z_1)^{k-1} \cdots (z - z_n)^{k_n},$$

where $z_1, \ldots, z_n$ are zeros of p and $k_1, \ldots, k_n \in \mathbb{N}$ are multiplicities of respective roots. Thus $k_1 + \cdots + k_n = k$.

22.1. Uniform convergence of holomorphic functions

Let f_n be holomorphic on $\mathcal{U}$ open. To show that a function f is holomorphic on $\mathcal{U}$, by Morera, we only need to verify that

$$\oint_{\partial R} f = 0$$

for any rectangle $R \subseteq \mathcal{U}$. Suppose $f_n \to f$ uniformly on each *compact* set in $\mathcal{U}$. Then given any rectangle $R \subseteq \mathcal{U}$, we have $f_n \to f$ uniformly on ∂R which is compact. Therefore, since each f_n is continuous on ∂R, so is f. Hence,

$$\oint_{\partial R} f_n \to \oint_{\partial R} f.$$

But then $\oint_{\partial R} f_n = 0$ by Goursat since each f_n is holomorphic, which forces $\oint_{\partial R} f = 0$. Since the claim holds true for every $R \subseteq \mathcal{U}$, f must be holomorphic on $\mathcal{U}$ by Morera. Therefore, we proved the analytic convergence theorem stated below.

Theorem 22.1 (Analytic convergence theorem)**.** *Let* $\{f_n\}$ *be a sequence of functions that are holomorphic on an open set* $\mathcal{U}$. *If* $f_n \to f$ *uniformly on compact subsets in* $\mathcal{U}$, *then* f *is holomorphic in* $\mathcal{U}$.

Corollary 22.1. *If* $f := \sum_{n=0}^{\infty} f_n$ *uniformly on compact subsets in* $\mathcal{U}$*, then* f *is holomorphic in* $\mathcal{U}$*.*

22.2. **On the convergence of derivatives**

Notice by Cauchy's inequality for $n = 1$, if g is holomorphic on $\mathcal{U}$ then

$$|g'(z)| \leq \frac{\|g\|_{C_r}}{r}.$$

If $K \subseteq \mathcal{U}$ is compact, then let $\varepsilon = \operatorname{dist}(K, \mathcal{U}^c) > 0$; and so for $0 < r < \varepsilon$, write $K_r := \bigcup_{z \in K} \overline{D(z,r)}$. One can verify that K_r is compact. So by Cauchy's inequality, for any $z \in K$ we have $\overline{D(z,r)} \subseteq \mathcal{U}$; therefore

$$|g'(z)| \leq \frac{\|g\|_{C_r(z)}}{r} \leq \frac{\|g\|_{K_r}}{r}.$$

Thus the above inequality holds for all $z \in K$, so

$$\|g'\|_K \leq \frac{1}{r}\|g\|_{K_r}.$$

Now suppose that f_n is holomorphic on $\mathcal{U}$ open, and $f_n \to f$ uniformly on compact sets in $\mathcal{U}$. Then f is holomorphic on $\mathcal{U}$ by the analytic convergence theorem.

Now consider any $K \subseteq \mathcal{U}$ compact. Then there is $r > 0$ such that

$$\|f_n' - f'\|_K \leq \frac{1}{r}\|f_n - f\|_{K_r} \to 0,$$

so $f_n' \to f'$ uniformly on compact sets.

Theorem 22.2 (Analytic convergence theorem II)**.** *Let* f_n *be holomorphic on an open set* $\mathcal{U}$*. If* $f_n \to f$ *uniformly on compact subsets of* $\mathcal{U}$*, then* $f_n' \to f'$ *uniformly on compact subsets in* $\mathcal{U}$ *also. Therefore,* $f_n^{(k)} \to f^{(k)}$ *uniformly on compact subsets in* $\mathcal{U}$ *for any* k*.*

Remark. Analytic convergence theorem II is *not* true at all in $\mathbb{R}$.

Corollary 22.2. *The analogous result holds for* $\sum f_n$*.*

22.3. **Reciprocal power series**

Consider the series

$$\sum_{n=1}^{\infty} b_n \frac{1}{z^n} = \sum_{n=1}^{\infty} b_n z^{-n}.$$

Suppose the power series $\sum b_n w^n$ has radius of convergence R. If $R = 0$, then $\sum_{n=1}^{\infty} b_n \left(\frac{1}{z}\right)^n$ diverges for all $z^{-1} \neq 0$, so the series diverges for all z. Suppose $R > 0$. Then $\sum b_n z^{-n}$ diverges for any $|z^{-1}| > R \Leftrightarrow |z| < R^{-1}$; similarly, $\sum b_n z^{-n}$ converges for $|z^{-1}| < R$ i.e., $|z| > R^{-1}$. Thus for any $0 < r < R$, we see that the series converges uniformly on $|z^{-1}| \leq r$, i.e., $|z| \geq r^{-1} > R^{-1}$.

We also know that

$$g(w) = \sum_{n=1}^{\infty} b_n w^n$$

is holomorphic on $D(0,R)$. So therefore, since

$$f(z) = g(z^{-1}) = \sum_{n=1}^{\infty} b_n \frac{1}{z^n}.$$

Let $A(a,b,c) := \{z \in \mathbb{C} : b < \operatorname{dist}(a,z) < c\}$. We claim then that f is holomorphic on $A(0, R^{-1}, \infty)$. Indeed, note that the change of variables from z to $1/z$ maps any point in $A(0, R^{-1}, \infty)$ onto $D(0,R)$, and g is holomorphic on $D(0,R)$. Therefore f is holomorphic on $A(0, R^{-1}, \infty) = \{z \in \mathbb{C} : |z| > R^{-1}\}$.

Definition 22.1. Given numbers $a_{-1}, a_{-2}, \dots$, we define

$$\sum_{n<0} a_n z^n := \sum_{n=-1,-2,\dots} a_n z^n = \sum_{n=1}^{\infty} a_{-n} z^{-n}.$$

Definition 22.2. A *Laurent series* about 0 is a series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n<0} a_n z^n + \sum_{n=0}^{\infty} a_n z^n = \cdots + a_{-2} z^{-2} + a_{-1} z + a_1 z + a_2 z^2 + \cdots$$

A Laurent series $\sum a_n z^n$ has two radii of convergence: one for $\sum_{n=0}^{\infty} a_n z^n$ and another for $\sum_{n=1}^{\infty} a_{-n} w^{-n}$. Call the respective radii of convergence R_+ and R_-. Thus the given Laurent series converges uniformly in $A(0, R_-^{-1}, R_+)$, and converges uniformly on $\overline{A(0,r,R)}$ for all $0 < R_+^{-1} < r < R < R_+$. It diverges outside of $\overline{A(0, R_-^{-1}, R_+)}$.

Inside $A(0, R_-^{-1}, R_+)$ the function

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

is holomorphic.

Definition 22.3. The annulus $A(0, R_-^{-1}, R_+)$ is called the *annulus of convergence* for the Laurent series.

Definition 22.4. A function f *has a Laurent expansion* at 0 if there exists an annulus $A(0, s, S)$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

for all $z \in A$ for some Laurent series that converges on A.

Proposition 22.1. *If a function has a Laurent series expansion on an annulus* $(0, r_1, r_2)$*, then such expansion is unique.*

Remark. If $R_- = \infty$ then $A(0, R_-^{-1}, R+ = D(0, R_+)$. If $R_+ = \infty$, ten $A(0, R_-^{-1}, R_+)$ is just $\{|z| > R_-^{-1}\}$. Finally, if $R_- = R_+ = \infty$ then $A(0, R_-^{-1}, R_+)$ is $\mathbb{C} \setminus \{0\}$.

Let $f(z) = \sum a_n z^n$ be a Laurent series. Then

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n z^{n-1} = \cdots + \frac{(-2)a_{-2}}{z^3} + \frac{(-1)a_{-1}}{z^2} + \frac{0}{z} + a_1 + 2a_2 z + \cdots .$$

Note that there is no z^{-1} term in $f'(z)$. In fact, both $f(z)$ and $f'(z)$ have the same annulus of convergence.

Notice that $\oint_{C_r(0)} \frac{dz}{z} = 2\pi i$. On the other hand, $\oint_{C_r(0)} z^m \, dz = 0$ for all $m \neq -1$ since z^n has a primitive on $\mathbb{C} \setminus \{0\}$. So

$$\oint_{C_r(0)} \frac{a_n z^n}{z^{k+1}} \, dz = \begin{cases} 2\pi i a_k & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}$$

This particular observation will be helpful in proving the uniqueness of the Laurent expansion. Suppose that f has Laurent expansion at 0. Then there exist s and S such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

for some annulus $s < |z| < S$. Now let $\gamma = C_r(0)$ for some $s < r < S$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

uniformly on γ, and so for each $k \in \mathbb{Z}$, we have

$$\frac{f(z)}{z^{k+1}} = \sum_{n=-\infty}^{\infty} a_n \frac{z^n}{z^{k+1}}$$

uniformly on γ (since $z^{-(k+1)}$ is bounded on γ). Hence

$$\oint_\gamma \frac{f(z)}{z^{k+1}} \, dz = \sum_{n=-\infty}^{\infty} \oint_\gamma \frac{a_n z^n}{z^{k+1}} \, dz = 2\pi i a_k.$$

Therefore for all k, a_k is completely determined by f, namely

$$a_k = \frac{1}{2\pi i} \oint_{C_r(0)} \frac{f(z)}{z^{k+1}} \, dz$$

for any r such that $s < r < S$.

Recall that to show that holomorphicity implies analyticity (more specifically, being holomorphic on a disc implies that that function has a power series expansion on that disc), we used the Cauchy integration formula, and the result regarding the Cauchy-type integrals. We proved this by applying

$$f(z) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{w - z} \, dw$$

and then by switching $\oint$ and $\sum$. We will carry out precisely the same operation at $a = 0$.

$$\frac{1}{w - z} = \frac{1}{w} \frac{1}{1 - \frac{z}{w}} = \frac{1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}},$$

and $\frac{1}{w-z}$ converges uniformly for $\left|\frac{z}{w}\right| \le \delta < 1$. So for f continuous on $C_r(0)$ and for each $z \in D(0,r)$,

$$\oint_{C_r} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty}\left(\oint_{C_r}\frac{f(w)}{w^{n+1}}\,dw\right)z^n.$$

Similarly, for $|z| > |w|$,

$$\frac{1}{w-z} = \frac{1}{z}\frac{1}{\frac{w}{z}-1} = \frac{-1}{z}\frac{1}{1-\frac{w}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{w^n}{z^{n+1}} = -\sum_{n=1}^{\infty}\frac{w^{n-1}}{z^n},$$

and the series converges uniformly for $|w/z| \le \delta < 1$. So for any f continuous on $C_r(0)$, we have

$$\oint_{C_r}\frac{f(w)}{w-z}\,dw = -\sum_{n=1}^{\infty}\left(\oint_{C_r} f(w)w^{n-1}\,dw\right)\frac{1}{z^n}$$

for each $|z| > r$. Notice that $w^{n-1} = (w^{-n+1})^{-1}$, so as long as f is continuous on C_r,

$$\oint_{C_r}\frac{f(w)}{w-z}\,dw = \begin{cases} \sum_{n=0}^{\infty}\left(\oint_{C_r}\frac{f(w)}{w^{n+1}}dw\right)z^n & \text{for } |z| < r \\ -\sum_{n<0}\left(\oint_{C_r}\frac{f(w)}{w^{n+1}}\,dw\right)z^n & \text{for } |z| > r, \end{cases}$$

thereby proving the uniqueness.

Theorem 22.3 (Laurent expansion theorem)**.** *Suppose that f is holomorphic on annulus $A(0,s,S) = \{z \in \mathbb{C} : s < |z| < S\}$. Then f has a Laurent series expansion on A.*

Proof. Let $s < r < R < S$, and consider the annulus $z \in A(0,r,R)$. For any $\theta \not\in \arg z$, let $\mu : [r,R] \to \mathbb{C}$ be $\mu(t) = te^{i\theta}$. Then (left as an exercise) $\gamma = C_R + \mu^- + C_r^- + \mu$ is homotopic to a point in A, and so $I(\gamma, z) = 1$. Since f is holomorphic on A, by the Cauchy integration formula,

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma}\frac{f(w)}{w-z}\,dw.$$

But since $\gamma = C_R + \mu^- + C_r - +\mu$ then

$$\oint_{\gamma} f = \oint_{C_R} f - \oint_{C_r} f.$$

So we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_R}\frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_r}\frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\oint_{C_R}\frac{f(w)}{w_{n+1}}\,dw\right)z^n - \frac{1}{2\pi i}\sum_{n<0}\left(\oint_{C_r}\frac{f(w)}{w^{n+1}}\,dw\right)z^n; \end{aligned}$$

observe that the first series expansion holds because $|z| < C_R$; and the second expansion follows since $|z| > r$. But then for each $n \in \mathbb{Z}$, we have $f(w)/w^{n+1}$ is holomorphic in A. Hence

$$\oint_C \frac{f(w)}{w^{n+1}}\,dw$$

is the same for every circle. So

$$f(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi i} \oint_C \frac{f(w)}{w^{n+1}}\, dw \right) z^n$$

for any circle C about 0 in A. Hence $f(z) = \sum a_n z^n$ for A where

$$a_n := \frac{1}{2\pi i} \oint_C \frac{f(w)}{w^{n+1}}\, dw$$

since the claim holds true for each $r < |z| < R$ for $s < r < R < S$. □

23. March 22

By translation all the same facts regarding the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

hold about any point $z_0 \in \mathbb{C}$. Also if a function f is holomorphic on an annulus $s < |z - z_0| < S$ then f has a unique Laurent expansion at z_0 valid in this annulus. Specifically, the coefficients are

$$a_n = \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}}\, dw$$

for any $s < r < S$.

Example. $f(z) = 1/(z(z-1))$ is holomorphic on $\mathbb{C} \setminus \{0, 1\}$, so f is holomorphic on the annulus $D^*(0,1) = A(0,0,1)$. Therefore we know that f has unique Laurent expansion about 0 in $D^*(0,1)$. For any $0 < |z| < 1$, the series

$$\begin{aligned} f(z) &= \frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} \\ &= -\frac{1}{z}\sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n-1} \\ &= -\frac{1}{z} - \sum_{n=1}^{\infty} z^{n-1} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n \\ &= -\frac{1}{z} - 1 - z - z^2 - \cdots, \end{aligned}$$

which is the Laurent expansion we are looking for. But f is holomorphic on everywhere except at the point $z = 0, z = 1$. We just examined what happens in $D^*(0,1)$, so now examine what happens outside of that region, namely $A(0,1,\infty)$ (i.e., $|z| > 1$). f has a unique Laurent expansion about 0 valid in this annulus due to holomorphicity. Hence

$$\begin{aligned} f(z) &= \frac{1}{z(z-1)} = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} \text{ for } |z| > 1 \\ &= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+2}} = \sum_{n=2}^{\infty}\frac{1}{z^n} \\ &= \cdots + z^{-4} + z^{-3} + z^{-2} \end{aligned}$$

is the valid Laurent expansion in $A(0,1,\infty)$.

24. March 22 & 25: Zeros and isolated singularities

Let g holomorphic on some disk $D(z_0,R)$ with zero at z_0. So g has power series expansion on $D(z_0,r)$; call this

$$g(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

Then from a previous assignment, we know that either $g\equiv 0$ on $D(z_0,r)$ or there is $k\in\mathbb{N}$ such that

$$g(z)=(z-z_0)^k\sum_{n=k}^{\infty}a_n(z-z_0)^n=(z-z_0)^kG(z)$$

where $G(z)$ is holomorphic and $G(z_0)\neq 0$, and $a_k\neq 0$.

Definition 24.1. The k as defined above is called the *order* or the *multiplicity* of the zero z_0 of g.

Definition 24.2. f has an *isolated singularity* at z_0 if f is holomorphic on some punctured disc D^* at z_0 and undefined at z_0. We know that f has a unique Laurent expansion

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

on D^* (annulus about z_0).

There are three types of isolated singularity z_0; we shall also characterize each type of singularity in terms of Laurent series.

24.1. Removable singularity

Definition 24.3. z_0 is a removable singularity if there exists a holomorphic g on some disc $D(z_0,r)$ such that $f=g$ on $D^*(z_0,r)$. Equivalently, there exists $\alpha\in\mathbb{C}$ such that for some $r>0$ we have

$$g(z):=\begin{cases}f(z) & (z\in D^*(z_0,r))\\ \alpha & (z=z_0)\end{cases}$$

is holomorphic. Note that in this case, our choice of α is unique due to continuity.

Proposition 24.1. *Let z_0 be a removable singularity of f. Then the following are equivalent.*

1. *z_0 has a removable singularity*
2. *The Laurent series for f at z_0 is a power series expansion, i.e.,*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

3. *$\lim_{z\to z_0}f(z)$ exists.*
4. *$\lim_{z\to z_0}(z-z_0)f(z)=0$.*
5. *f is bounded near z_0.*

24.2. **Pole**

Definition 24.4. z_0 is a *pole* of f if there exists a holomorphic function g on some $D(z_0, r)$ with a zero at z_0 so that $f = \frac{1}{g}$ on $D^*(z_0, r)$. Equivalently, for some $r > 0$ the function

$$g(z) := \begin{cases} \frac{1}{f(z)} & (z \in D^*(z_0, r)) \\ 0 & (z = z_0) \end{cases}$$

is holomorphic on $D(z_0, r)$.

Notice that $g \not\equiv 0$ on $D(z_0, r)$ so z_0 is a zero for g with finite order $k \in \mathbb{N}$. Hence we know that $g(z) = (z - z_0)^k G(z)$ on $D(z_0, r)$ where $G(z)$ is holomorphic and never zero on $D(z_0, r)$. Hence

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^k} \frac{1}{G(z)}.$$

$1/G(z)$ is never 0 and is holomorphic on $D(z_0, r)$, so $1/G(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$ with $c_0 \neq 0$. All in all,

$$f(z) = \frac{c_0}{(z - z_0)^k} + \cdots + \frac{c_{k-1}}{z - z_0} + c_k + c_{k+1}(z - z_0) + \cdots .$$

Define $c_{i+k} = a_i$ for all $i > -k$; then we see that the Laurent series for f has all $a_n = 0$ for $n < -k$ and $a_{-k} \neq 0$.

Definition 24.5. The k above is called the *order* or *multiplicity* of the pole z_0; $-k$ is the *integer order of the pole* z_0, and we write $-k := \operatorname{ord}(f, z_0)$. If $k = 1$, then we call z_0 a *simple pole*.

Proposition 24.2. *Let z_0 be a pole of order k in f. Then the following are equivalent.*

(1) *z_0 is a pole of order k*

(2) *$f = 1/g$ on some $D^*(z_0, r)$, where g is holomorphic on $D(z_0, r)$ with zero of order k at z_0.*

(3) *Laurent series of f at z_0 has the form*

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

on D^* where $a_{-k} \neq 0$.

(4) *$f(z)$ is of the form*

$$f(z) = \frac{1}{(z - z_0)^k} G(z)$$

on some $D^*(z_0, r)$, where G is holomorphic and is non-zero on $D(z_0, r)$.

(5) *$(z - z_0)^k f(z)$ has a non-zero removable singularity at z_0.*

(6) *$\lim_{z \to z_0} (z - z_0)^k f(z)$ exists and is non-zero.*

(7) *$f \neq 0$ near z_0, and*

$$\lim_{z \to z_0} f(z) = \infty,$$

or equivalently,

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0.$$

Remark. Suppose that f is holomorphic and is non-zero on D^*. Thus, $1/f$ is holomorphic and is non-zero as well. Hence, f has a non-zero removable singularity at z_0 if and only if $1/f$ has a non-zero removable singularity at z_0. Similarly, f has a removable singularity at z_0 that is a zero of order k if and only if $1/f$ has a pole of order k at z_0. Hence, we can conclude that f has a removable singularity at z_0 if and only if $1/f$ has a pole or a non-zero removable singularity at z_0.

Furthermore, notice that f is bounded away from 0 near z_0 if and only if there is $\varepsilon > 0$ such that $|f(z)| \geq \varepsilon$ on some $D^*(z_0, r)$. This is also equivalent to saying that $f \neq 0$ and $1/|f(z)| < 1/\varepsilon$ on some $D^*(z_0, r)$; also equivalently, $1/f$ has removable singularity at z_0 – and this happens if and only if f has a pole or non-zero removable singularity.

24.3. **Essential singularity**

Definition 24.6. Let f be holomorphic on D^*, and be undefined at z_0. Then z_0 is an *essential singularity* for f if the singularity at z_0 is not removable, nor is it a pole.

Proposition 24.3. *Let z_0 be an essential singularity in f. Then the following are equivalent.*

(1) z_0 is an essential singularity in f.
(2) The Laurent series of f on D^* has infinitely many terms of negative index, i.e.,

$$f(z) = \cdots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

(3) Neither of the following limits exist:

$$\lim_{z \to z_0} f(z), \lim_{z \to z_0} \frac{1}{f(z)}.$$

Definition 24.7. Let z_0 be an essential singularity in f, and that $f(z)$ has the Laurent expansion

$$f(z) = \cdots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then the *residue* of f $\operatorname{Res}(f, z_0)$ is the value of a_{-1}. The *principal part* of the Laurent expansion of f is

$$\cdots + \frac{a_{-1}}{z - z_0} = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n.$$

We will examine what happens at essential singularities. Suppose that f has an essential singularity at z_0. Then so does $f(z) - \alpha$ for any $\alpha \in \mathbb{C}$ since $f(z) - \alpha$ and $f(z)$ have the same principal part. Also, z_0 is not a pole or a removable singularity for $f(z) - \alpha$. $f(z) - \alpha$ is not bounded away from 0 near z_0. Hence, $f(z)$ is not bounded away from α near z_0. This gives rise to the following theorem.

Theorem 24.1 (Casorati-Weierstrass theorem)**.** *If f has an essential singularity at z_0 then for any $\alpha \in \mathbb{C}$, there exists $z_n \to z_0$ such that $f(z_n) \to \alpha$.*

24.4. **Residue theory**

Suppose that f has an isolated singularity at z_0, i.e., f is holomorphic on some $D^*(z_0, r)$. Then f has Laurent expansion on $D^*(z_0, r)$; say this Laurent expansion is

$$f(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Let $S(z)$ be the singular part of the Laurent series, i.e.,

$$S(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n},$$

the reciprocal power series with $R^- = \infty$ converges for $|z - z_0| > 1/R^- = -0$ (i.e., holomorphic on $\mathbb{C} \setminus \{0\}$). Hence, $S(z)$ converges uniformly on each $\overline{A(z_0, \varepsilon, \infty)} = \{z : |z - z_0| \geq \varepsilon\}$.

Suppose that γ is any closed curve in $\mathbb{C} \setminus \{z_0\}$. Then for $n \neq 1$, we have

$$\oint_\gamma \frac{dz}{(z - z_0)^n} = 0,$$

whereas

$$\oint_\gamma \frac{dz}{z - z_0} = 2\pi i I(\gamma, z_0).$$

Since $z_0 \in \mathbb{C} \setminus \gamma$, and $\mathbb{C} \setminus \gamma$ is open, there exists a disc $D(z_0, r) \subset \mathbb{C} \setminus \gamma$. Therefore $\gamma \subset \overline{A(z_0, r, \infty)}$. Thus the reciprocal power series converges uniformly on γ. In conclusion,

$$\oint_\gamma S(z)\, dz = \sum_{n=1}^{\infty} \oint_\gamma \frac{a_{-n}}{(z - z_0)^n}\, dz = a_{-1}(2\pi i) I(\gamma, z_0) = \operatorname{Res}(f, z_0) 2\pi i I(\gamma, z_0).$$

Theorem 24.2 (Residue theorem)**.** *Let $\mathcal{U}$ be an open set, and $\gamma \sim p$ in $\mathcal{U}$ for some point $p \in \mathcal{U}$. Let $z_1, \ldots, z_n \in \mathcal{U} \setminus \gamma$. If f is holomorphic on $\mathcal{U} \setminus \{z_1, \ldots, z_n\}$, then*

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j) I(\gamma, z_j).$$

Proof. Let $\mathcal{U} \setminus \gamma$ be open. For each z, there exists a disc $D(z_j, r_j) \subset \mathcal{U} \setminus \gamma$ such that $z_k \notin D(z_j, r_j)$ for $k \neq j$. So f is holomorphic on each punctured disc $D^*(z_j, r_j)$, so f has Laurent expansion

$$f(z) = S_j(z) + g_j(z)$$

where $S_j(z)$ is the singular part which is holomorphic on $\mathbb{C} \setminus \{z_j\}$ and g_j the power series part which is holomorphic on $D(z_j, r_j)$. Then as we already argued previously, we have

$$\oint_\gamma S_j = 2\pi i \operatorname{Res}(f, z_i) I(\gamma, z_i).$$

Let

$$g(z) = f(z) - \sum_{j=1}^{n} S_j(z).$$

Note that $f(z)$ is holomorphic on $\mathcal{U} \setminus \{z_1, \ldots, z_n\}$; on the other hand $\sum S_j(z)$ is holomorphic on everywhere except for $\{z_1, \ldots, z_n\}$. Thus $g(z)$ is holomorphic on $\mathcal{U} \setminus \{z_1, \ldots, z_n\}$. But on each $D^*(z_k, r_k)$, we have

$$g(z) = f(z) - S_k(z) - \sum_{j \neq k} S_j(z).$$

But note that $g_k(z) := f(z) - S_k(z)$ is holomorphic on $D(z_k, r_k)$; similarly, $\sum_{j \neq k} S_j(z)$ is holomorphic on $D(z_k, r_k)$ as well, meaning that g in fact has a removable singularity at the z_j's. Now we can apply Cauchy's theorem to get

$$\oint_\gamma g = 0.$$

The theorem now follows. □

25. March 27

Definition 25.1. f is *meromorphic* on an open set $\mathcal{U}$ if f is holomorphic on $\mathcal{U}$ except at points that are poles. If P is the set of poles of f in $\mathcal{U}$, then P is a set of isolated points, since isolated singularities are isolated by default. Thus P is discrete, and at most countable.

Proposition 25.1. *P cannot accumulate in $\mathcal{U}$. Therefore, P has no limit points in $\mathcal{U}$.*

Let f be meromorphic on $\mathcal{U}$ an open set. So at each $z_0 \in \mathcal{U}$, f is either holomorphic or has a pole at z_0, so there exists $k \in \mathbb{Z}$ such that (with $a_k \neq 0$)

$$\begin{aligned} f(z) &= a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots \\ &= a_k(z - z_0)^k \left(1 + \frac{a_{k+1}}{a_k}(z - z_0) + \cdots\right) = a_k(z - z_0)^k g(z) \end{aligned}$$

on some $D(z_0, r)$; evidently, $g(z)$ is holomorphic on $D(z_0, r)$. If $k < 0$, then z_0 is a pole of order $-k$ (or integer order k); if $k = 0$, then f is holomorphic, and is $\neq 0$ at z_0; if $k > 0$, then z_0 is a zero of order k. Furthermore, $g \neq 0$ on $D(z_0, r)$ unless $f \equiv 0$ on connected components of $\mathcal{U} \ni z_0$.

Recall that $(gh)' = gh' + g'h$ and $(hg)'/(hg) = h'/h + g'/g$. Thus

$$\frac{f'}{f} = \frac{ka_k(z - z_0)^{k-1}}{a_k(z - z_0)^k} + \frac{g'}{g} = \frac{k}{z - z_0} + \frac{g'}{g}$$

on $D^*(z_0, r)$. Since $g \neq 0$ on $D(z_0, r)$ and is holomorphic, it follows that g'/g is holomorphic on $D(z_0, r)$. Hence this is the Laurent series of f'/f. Thus

$$\operatorname{Res}\left(\frac{f'}{f}, z_0\right) = k,$$

the integer order of pole at z_0, or order of zero of f at z_0; or $k = 0$ (i.e., f is holomorphic and is non-zero at z_0). Since f is meromorphic on an open set $\mathcal{U}$ and not zero on any component of $\mathcal{U}$, it follows that f'/f is holomorphic on $\mathcal{U}$ except for simple poles at the zeros and poles of f. And for each zero and pole z of f, indeed $\operatorname{Res}(f'/f, z) = \operatorname{ord}(f, z)$, the integer order of a zero or a pole.

Theorem 25.1. *Suppose that $\mathcal{U}$ is an open set, and a curve $\gamma \sim p$ for some point $p \in \mathcal{U}$. Let f be a meromorphic function on $\mathcal{U}$, and has zeros and poles at $z_1, \dots, z_n \in \mathcal{U} \setminus \gamma$. Then*

$$\oint_\gamma \frac{f'}{f} = 2\pi i \sum_{j=1}^{n} \operatorname{ord}(f, z_j) I(\gamma, z_j).$$

Proof. Since f'/f is holomorphic on $\mathcal{U} \setminus \{z_1, \dots, z_n\}$, the residue theorem implies that

$$\oint_\gamma \frac{f'}{f} = 2\pi i \sum_{j=1}^{n} \operatorname{Res}\left(\frac{f'}{f}, z_j\right) I(\gamma, z_j).$$

The claim follows upon realizing that $\operatorname{Res}(f'/f, z_j) = \operatorname{ord}(f, z_j)$. □

Remark. Recall that via change of variables,

$$\oint_\gamma \frac{f'}{f} = \oint_{f\circ\gamma} \frac{1}{z}\, dz = 2\pi i I(f \circ \gamma, 0) = i(\Delta \arg f \circ \gamma),$$

where $\Delta \arg f \circ \gamma$ denotes the change in argument over the curve $f \circ \gamma$. Thus by the argument principle,

$$\frac{1}{2\pi} \Delta \arg f \circ \gamma = I(f \circ \gamma, 0) = \sum_{j=1}^{n} \operatorname{ord}(f, z_j) I(\gamma, z_j).$$

Of our particular interest is when γ is a simple closed curve since its winding number can be easily characterized, specifically

$$I(\gamma, z_j) = \begin{cases} 1 & \text{if } z_j \text{ inside } \gamma \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$\frac{\Delta \arg f \circ \gamma}{2\pi} = I(f \circ \gamma, 0) = n_z - n_p,$$

where n_z denotes the number of zeros of f inside γ and n_p the number of poles of f inside γ (all counted with multiplicities – that is, a pole of order m counts as $-m$). Hence, we have

$$\oint_\gamma \frac{f'}{f} = 2\pi i (n_z - n_p).$$

25.1. **Application of argument principle**

We now discuss some applications of the argument principle. The argument principle can be used to count zeros. As usual, suppose that $\mathcal{U}$ is open, and γ is simple such that $\gamma \sim p$ for some point $p \in \mathcal{U}$. If f is holomorphic on $\mathcal{U}$ and non-zero for every $z \in \gamma$, then $\Delta \arg(f \circ \gamma) = 2\pi n_z$. Thus you can compute $\Delta \arg f \circ \gamma$ to compute n_z.

Now suppose that f and g are holomorphic on $\mathcal{U}$ open. If $F = g/f$, then

$$\frac{F'}{F} = \frac{f}{g} \frac{g'f - gf'}{f^2} = \frac{g'}{g} - \frac{f'}{f}$$

on $\mathcal{U} \setminus Z$, where Z is the set of zeros of f and g. Hence

$$\oint_\gamma \frac{F'}{F} = \oint_\gamma \frac{g'}{g} - \oint_\gamma \frac{f'}{f}$$

for any simple closed curve γ in $\mathcal{U}$ such that no zeros of f and g are on γ. Note that

$$I(F \circ \gamma, 0) = I(g \circ \gamma, 0) - I(f \circ \gamma, 0),$$

so if $I(F \circ \gamma, 0) = 0$ then $I(g \circ \gamma, 0) = I(f \circ \gamma, 0)$. Hence f and g have the same number of zeros inside any simple closed curve homotopic to a point in $\mathcal{U}$.

Example. If $F \circ \gamma$ lies in $D(1, 1)$, then $I(F \circ \gamma, 0) = 0$. $|1 - F(z)| < 1$ for z on γ, i.e.,

$$\left|1 - \frac{g(z)}{f(z)}\right| < 1 \Rightarrow |f(z) - g(z)| < |f(z)|$$

on γ.

Theorem 25.2 (Rouché's theorem). *Let $\mathcal{U}$ be open, and γ a simple closed curve homotopic to a point in $\mathcal{U}$. If f, g are both holomorphic on $\mathcal{U}$ with $|f(z) - g(z)| < |f(z)|$ on γ, then f and g both have the same number of zeros inside γ.*

26. March 29

26.1. Mapping theorem

Suppose that f is holomorphic on an open set $\mathcal{U}$. For any $z_0 \in \mathcal{U}$, "$f(z_0) = w_0$ with order k" means "$f(z) - w_0$ has a zero of order k at z_0". Thus, a power series of $f - w_0$ starts at order k. That is, $f(z)$ is of the form

$$f(z) = w_0 + a_k(z - z_0)^k + \cdots +,$$

i.e., $f(z_0) = w_0, f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$, but $f^{(k)}(z_0) \neq 0$. Let f be holomorphic on $\mathcal{U}$ open and $z_0 \in \mathcal{U}$, and $f(z_0) = w_0$ with order k. So $F := f - w_0$ is holomorphic on $\mathcal{U}$ with zero of order k at z_0. So there exists $r > 0$ such that $F \neq 0$ on $D^*(z_0, r)$.

Let $0 < \delta < r$ and $\gamma = C_\delta(z_0)$. Then F is continuous at and is non-zero on a compact γ, so $|F(z)| \geq \varepsilon > 0$ on γ for some $\varepsilon > 0$, i.e., $|f(z) - w_0| \geq \varepsilon$ on γ.

Therefore, for any $w \in D(w_0, \varepsilon)$, we have $|w - w_0| < \varepsilon \leq |F(z)|$ for all $z \in \gamma$. Define $G(z) := f(z) - w$. Then by Rouché's theorem, F and G have the same number of zeros inside γ, i.e., in $D(z_0, \delta)$. But F is zero only at z_0, with order k. Therefore, G has k zeros in $D(z_0, \delta)$. Thus, f sends k points in $D(z_0, \delta)$ (counting multiplicities) to w.

In particular, there is $z \in D(z_0, \delta)$ such that $f(z) = w$ – note this is true for all $w \in D(w_0, \varepsilon)$. Hence, for all small enough discs $D(z_0, \delta)$ there is some $D(f(z_0), \varepsilon) \subset f(D(z_0, \delta))$.

Definition 26.1. A map f is *open* if $\mathcal{U}$ being open implies $f(\mathcal{U})$ is also open.

Proposition 26.1. *f is open if and only if for any small enough $D(z_0, \delta)$ there exists some $D(f(z_0), \varepsilon) \subset f(D(z_0, \delta))$.*

Theorem 26.1 (Open mapping theorem). *Suppose that $\mathcal{U}$ is open and f is holomorphic on $\mathcal{U}$ and is non-constant on each component of $\mathcal{U}$. If $f(z_0) = w_0$ with order k, then f is "locally k-to-one (counting multiplicities) near z_0. In other words, for all small enough disc $D(z_0, \delta)$, there exists a disc $D(w_0, \delta)$ such that each value $w \in D(w_0, \varepsilon)$ is achieved exactly k times (counting multiplicities) in $D(z_0, \delta)$.*

26.2. **Some applications of the open mapping theorem**

Suppose that $\mathcal{U}$ is open, and f is holomorphic on $\mathcal{U}$ and is locally one-to-one (i.e., f is one-to-one on a neighbourhood of each point in $\mathcal{U}$). Suppose that $f'(z_0) = 0$ for some $z_0 \in \mathcal{U}$. Then z_0 must be an isolated zero of f'. Otherwise, $f' \equiv 0$ near z_0, so $f \equiv C$ for some constant C, but this contradicts that f is one-to-one near z_0. Thus $f' \neq 0$ on some $D^*(z_0, r)$. Let $f(z_0) = w_0$. Then $f(z_0) = w_0$ with order ≥ 2 (because $f'(z_0) = 0$). So by the open mapping theorem, for all small discs $D \ni z_0$, there exists a disc $D' \ni w_0$ such that each $w \in D'$ is at least achieved twice in D. However, there could be z with order 2 such that $f(z) = w$. But for $D \subset D(z_0, r)$ we have $f' \neq 0$ at all $z \neq z_0$. Thus f must have order 1 at each $z \neq z_0$. But f is not one-to-one on a sufficiently small disc D, but this contradicts the fact that f is locally one-to-one. Note that this proves one direction ($\Rightarrow$) of the following theorem, with the other directions following from the inverse function theorem.

Theorem 26.2. *If f is holomorphic on an open set $\mathcal{U}$, then f is locally one-to-one on $\mathcal{U}$ if and only if $f' \neq 0$ in $\mathcal{U}$.*

Definition 26.2. If f satisfies the conditions outlined in the above theorem, then f is said to be *conformal*.

Note that if $\mathcal{U}$ is open, then so is $f(\mathcal{U})$ by the open mapping theorem. Also, if $f : \mathcal{U} \to \mathcal{V}$ is holomorphic and invertible, then $f' \neq 0$ on $\mathcal{U}$. Then by the inverse function theorem, $f^{-1} : \mathcal{V} \to \mathcal{U}$ is holomorphic. In this case, we say that f is *biholomorphic*. Furthermore, $\mathcal{U}$ and $\mathcal{V}$ are said to be *conformally equivalent*, and we also say that f *maps* $\mathcal{U}$ *conformally onto* $\mathcal{V}$. Hence, if f is biholomorphic, then both f and f^{-1} are conformal.

27. April 1: More applications of the open mapping theorem

27.1. **Maximum modulus principle**

Suppose that $\mathcal{U}$ is open, and f is holomorphic on $\mathcal{U}$. Suppose that $|f|$ has a local maximum at $z_0 \in \mathcal{U}$, i.e., $|f(z_0)| \geq |f(z)|$ for all $z \in D(z_0, r)$ for some r. Then since $D := D(z_0, r)$ is open, by the open mapping theorem, $f(D)$ is open unless $f \equiv C$ for some constant C on D. Therefore, there exists $z \in D$ such that $|f(z_0)| < |f(z)|$, but this is a contradiction since z_0 is the point of local maximum. This proves the maximum modulus principle stated below.

Theorem 27.1 (Maximum modulus principle). *If f is holomorphic and $\mathcal{U}$ is an open set, then $|f|$ cannot have a local maximum in $\mathcal{U}$ except when f is a constant on that components of $\mathcal{U}$.*

Remark. If K is compact subset of $\mathcal{U}$, K° is a domain, and f is holomorphic on $\mathcal{U}$, then $|f|$ is continuous on compact K. Therefore $|f|$ must have a maximum value at K. But by the maximum modulus principle, the maximum cannot be achieved in K° unless f is constant on K°. Hence either f is constant on K by continuity, or if there is the maximum, it must occur on ∂K, the boundary of K. This yields the following inequality, for all $z \in K^\circ$:

$$|f(z)| \leq \sup_{w \in \partial K} |f(w)|.$$

Since $z \in K^\circ$, if the equality is achieved at some point z, then the equality must be achieved at all points. Hence in this case f is constant.

27.2. **Application of the maximum modulus principle: Schwarz lemma**

Let $D = D(0,1)$. Suppose that f is holomorphic on D, and $f(D) \subset D$, with $f(0) = 0$. Since f is holomorphic on disc D, it has a power series expansion on D. So if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

on D, then $a_0 = 0$ since $f(0) = 0$. Hence $f(z) = a_1 z + a_2 z^2 + \cdots = z(a_1 + a_2 z + \cdots)$, so the function

$$\frac{f(z)}{z} = a_1 + a_2 z + \cdots$$

has a removable singularity at $z = 0$. Thus the function

$$g(z) = \begin{cases} \frac{f(z)}{z} & (z \in D^*) \\ a_1 = f'(0) & (z = 0) \end{cases}$$

is holomorphic on D. Suppose $0 < r < 1$, and write $D_r := D(0,r)$. Then by the maximum modulus principle, we have

$$|g(z)| \leq \sup_{|w|=r} |g(w)| = \sup_{|w|=r} \frac{|f(w)|}{|w|} = \frac{1}{r} \sup_{|w|=r} |f(w)|.$$

But note that since $f(D) \leq D$, $\sup |f(w)|$ is bounded above at 1. So we have

$$|g(z)| \leq \frac{1}{r}$$

on D_r. So for each $z \in D$, we have $z \in D_r$ for all $|z| < r < 1$. Thus $|g(z)| \leq r^{-1}$ for all $|z| < r < 1$. Hence $|g(z)| \leq s = r^{-1}$ for all $s > 1$. This implies taht $|g(z)| \leq 1$ for all $z \in D$. This means that by definition of g, we have $|f(z)| \leq |z|$ for all $z \in D^*$ and $|f'(0)| = |g(0)| \leq 1$. Furthermore, if $|g(z_0)| = 1$ for some $z_0 \in D$ (i.e., if $|f(z_0)| = |z_0|$ for some $z_0 \in D^*$ or if $|f'(0)| = 1$, then $|g|$ has a maximum at $z_0 \in D$; so $g \equiv C$ for some constant on D. Hence $f(z)/z \equiv C$ on D^*, or $f(z) = cz$ on D (which is trivially true at $z = 0$). But then $1 = |g(z_0)| = |c|$, so $|c| = e^{i\theta}$ for some θ. In conclusion, $f(z) = e^{i\theta} z$. In other words, $f(z)$ must be a rotation R_θ by θ. This proves the Schwarz lemma, which is formally stated below.

Theorem 27.2 (Schwarz lemma)**.** *Suppose that $f : D(0,1) \to D(0,1)$ is holomorphic, and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in D(0,1)$. If the equality holds at some $z_0 \neq 0$, then f is a rotation. In particular, $|f'(0)| \leq 1$; if the equality holds, then f is a rotation.*

Proposition 27.1. *If $f : D(0,1) \to D(0,1)$ is holomorphic, bijective, and $f(0) = 0$, then f is a rotation.*

Proof. Exercise. (*Hint:* You can apply the Schwarz lemma to both f and f^{-1}. Think about why this is the case.) □

Remark. Conversely, every rotation $R_\theta(z) = e^{i\theta} z$ is a biholomorphic map from $D(0,1)$ to itself that fixes 0.

Thus this gives rise to the following definition.

Definition 27.1. A biholomorphic map $f : \mathcal{U} \to \mathcal{U}$ is called an *analytic automorphism* of $\mathcal{U}$, and we write $\operatorname{Aut}(\mathcal{U})$ to represent the set of all analytic automorphisms of $\mathcal{U}$.

So by the proposition above, f is an analytic automorphism with $f(0) = 0$ if and only if f is a rotation R_θ.

Example. For any $\alpha \in D(0,1)$,

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Note ψ_α is holomorphic on $\mathbb{C} \setminus \{\overline{\alpha}^{-1}\}$, and ψ_α never takes the value $\overline{\alpha}^{-1}$ (left as an exercise). Also, $\psi_\alpha(\psi_\alpha(z)) = z$, so ψ_α is invertible with $\psi_\alpha^{-1} = \psi_\alpha$, which implies

$$\psi_\alpha \in \operatorname{Aut}\left(\mathbb{C} \setminus \left\{\frac{1}{\overline{\alpha}}\right\}\right)$$

Also ψ_α sends 0 to α and α to 0, and $|\psi_\alpha(z)| = 1$ for all $|z| = 1$.

27.3. Automorphisms of the unit disc

Let $D = D(0,1)$. We now that $\{f \in \operatorname{Aut}(D) : f(0) = 0\} = \{R_\theta : \theta \in \mathbb{R}\}$. Also, for each $\alpha \in D$,

$$\psi_\alpha : \mathbb{C} \setminus \left\{\frac{1}{\overline{\alpha}}\right\} \to \mathbb{C} \setminus \left\{\frac{1}{\overline{\alpha}}\right\}$$

is biholomorphic, and in particular the domain contains $\overline{D}$ with $|\psi_\alpha| \equiv 1$ for all ∂D. Therefore $|\psi_\alpha(z)| < 1$ for all $z \in D$ by the maximum modulus principle because ψ_α is invertible and hence cannot be constant. So $\psi_\alpha(D) \subset D$, and this implies $\psi_\alpha \in \operatorname{Aut}(D)$.

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Recall that $\operatorname{Aut}\mathcal{U}$ for an open set $\mathcal{U}$ denotes the set of all analytic automorphisms of $\mathcal{U}$. Note that if $f : \mathcal{U} \to \mathcal{V}$ and $g : \mathcal{V} \to \mathcal{W}$ are both biholomorphic, then so is $g \circ f : \mathcal{U} \to \mathcal{W}$.

Proposition 28.1. $\operatorname{Aut}\mathcal{U}$ *is a group under composition. Furthermore, if* $g : \mathcal{U} \to \mathcal{V}$ *is biholomorphic then* $\operatorname{Aut}\mathcal{U} \cong \operatorname{Aut}\mathcal{V}$. *Specifically, the map* $\eta : \operatorname{Aut}\mathcal{U} \to \operatorname{Aut}\mathcal{V}$ *defined by* $\varphi \mapsto g \circ \varphi \circ g^{-1}$ *is a group isomorphism.*

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\varphi} & \mathcal{U} \\ {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle g^{-1}} \\ \mathcal{V} & \xrightarrow{g \circ \varphi \circ g^{-1}} & \mathcal{V} \end{array}$$

Therefore, if we know $\operatorname{Aut}\mathcal{U}$*, then we k now* $\operatorname{Aut}\mathcal{V}$ *for any conformally equivalent open set* $\mathcal{V}$*.*

28.1. On $\operatorname{Aut} D$

Particularly, if $D = D(0,1)$ is a unit disc, we saw that $\{\varphi \in \operatorname{Aut} D : \varphi(0) = 0\} = \{R_\theta : \theta \in \mathbb{R}\}$, and furthermore for any $\alpha \in D$, if

$$\psi_a(z) = \frac{\alpha - z}{1 - \overline{\alpha} z} \in \operatorname{Aut} D$$

such that $\psi_\alpha^{-1} = \psi_\alpha$. Also ψ_α switches 0 and α. Let's see if there are other analytic automorphisms of D. Let $f \in \operatorname{Aut} D$, and suppose that $\alpha = f^{-1}(0)$. So $f \circ \psi_\alpha \in \operatorname{Aut} D$ fixes 0, so $f \circ \psi_\alpha = R_\theta$ for some θ, so hence $f = R_\theta \circ \psi_\alpha$. Also, for all α and θ, we have $R_\theta \in \psi_\alpha \in \operatorname{Aut} D$. Thus putting all these observation together gives the following result.

Proposition 28.2. *$f \in \operatorname{Aut} D$ if and only if*

$$f(z) = e^{i\theta}\frac{\alpha - z}{1 - \overline{\alpha}z}$$

for some $\theta \in \mathbb{R}$ and $\alpha := f^{-1}(0) \in D$.

28.2. **Conformal equivalence to the unit disc**

If $\mathcal{U}$ and $\mathcal{V}$ are conformally equivalent (i.e., there exists a biholomorphic map $f : \mathcal{U} \to \mathcal{V}$), then the continuity of f implies that $f(\mathcal{U}) = \mathcal{V}$ must be connected provided $\mathcal{U}$ is connected also. Similarly, since f is a homeomorphism, one can argue that if $\mathcal{U}$ is simply connected, then $f(\mathcal{U}) = \mathcal{V}$ is simply connected as well. Finally, if $\mathcal{U}$ is bounded, then $\mathcal{V} \neq \mathbb{C}$ thanks to Liouville's theorem. We thus can conclude that if $\mathcal{U}$ is conformally equivalent to a unit disc, then $\mathcal{U}$ must be proper (i.e., $\mathcal{U} \neq \mathbb{C}$), non-empty, simply connected domain. While we do not have enough time to fully prove the following theorem, we nonetheless state the theorem below.

Theorem 28.1 (Riemann mapping theorem)**.** *If $\mathcal{U}$ is a proper, non-empty, simply connected domain, then $\mathcal{U}$ is conformally equivalent to the unit disc.*

Note that each $f \in \operatorname{Aut} D$ is of the form

$$f(z) = e^{i\theta}\frac{\alpha - z}{1 - \overline{\alpha}z},$$

and that $f(z)$ is in fact a map of the form $(az + b)/(cz + d)$.

Definition 28.1. A *fractional linear transformation* (or *Möbius transformation*) is a non-constant map of the form

$$f(z) = \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

If $c = 0$, then $f(z) = (a/d)z + (b/d) : \mathbb{C} \to \mathbb{C}$ is entire and invertible by inverse

$$f^{-1}(z) = \frac{d}{a}z - \frac{b}{a}.$$

If $c \neq 0$, then f is holomorphic on $\mathbb{C} \setminus \{-d/c\}$ and never takes the value a/c. In fact,

$$\frac{az + b}{cz + d} = w \Leftrightarrow z = \frac{dw - b}{-cw + a}.$$

Therefore, $f : \mathbb{C} \setminus \{-d/c\} \to \mathbb{C} \setminus \{a/c\}$ is biholomorphic with a fractional linear inverse.

Remark. We can extend the definition of $f(z) = (az+b)/(cz+d)$ to $f : \mathbb{C} \setminus \{\infty\} \to \mathbb{C} \setminus \{\infty\}$ by letting $f(\infty) = \infty$ if $c = 0$; let $f(-d/c) = \infty$ and $f(\infty) = a/c$ if $c \neq 0$. Then $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is bijective.

Proposition 28.3. *The set of fractional linear transformations forms a group.*

Proposition 28.4. *Given distinct points $p_1, p_2, p_3 \in \mathbb{C} \cup \{\infty\}$ and $q_1, q_2, q_3 \in \mathbb{C} \cup \{\infty\}$, there exists a* unique *fractional linear transformation f such that $f(p_j) = q_j$ for $j = 1, 2, 3$.*

Note that there exists a unique circle or a unique line through any three distinct points.

28.3. **Matrices and fractional linear transformations**

Notice that each matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$$

defines a fractional linear transformation

$$f_A(z) = \frac{az+b}{cz+d}.$$

Note that this correspondence is not one-to-one, since

$$\frac{az+b}{cz+d} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d}$$

for any $\lambda \neq 0$.

Proposition 28.5. *The above correspondence defines a group homomorphism. In other words, we have*

$$f_{AB} = f_A \circ f_B; f_{A^{-1}} = f_A^{-1}; \text{ and } f_A = \mathrm{id} \Leftrightarrow A = \lambda I, \lambda \neq 0.$$

Now let's look at some special cases.

Example. The Möbius transformation corresponding to

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

is the translation map, namely $z + b$. The scaled rotation az corresponds to a matrix

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Note also that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

corresponds to the inversion map $1/z$.

In fact, every fractional linear transformation $f(z) = (az + b)/(cz + d)$ is a composition of the maps of the above types. Therefore a fractional linear transformation sends lines or circles into lines or circles.

What if $a, b, c, d \in \mathbb{R}$ rather than in $\mathbb{C}$? In this case, both $-d/c$ and a/c are real numbers, so

$$\mathrm{Im}(f(z)) = \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc}{|cz+d|^2}\mathrm{Im}(z).$$

Thus if $ad - bc > 0$ then $\mathrm{Im}(f(z)) > 0$ if and only if $\mathrm{Im}(z) > 0$. But then any

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

with a positive determinant is

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant 1 for some $\lambda \in \mathbb{R}$. Hence, for each $A \in \mathrm{SL}(2, \mathbb{R})$, we have $f_A \in \mathrm{Aut}\, H$ where H is the upper half plane.

28.4. **Cayley transform**

Definition 28.2. The *Cayley transform* is the fractional linear transformation

$$C(z) = \frac{i-z}{i+z} = \frac{-z+i}{z+i}$$

with

$$C^{-1}(z) = \frac{iz-i}{-z-1} = i\frac{1-z}{1+z}.$$

Notice that $C : \mathbb{C} \setminus \{-i\} \to \mathbb{C} \setminus \{-1\}$ satisfies $|C(z)| = 1$ if and only if $|i-z| = |i+z|$, which is equivalent to saying that z is equidistant form both i and $-i$. Therefore, z must be real. Also, $z \in H \Leftrightarrow |i-z| < |i+z|$ which is true if and only if z is close to i than to $-i$, and this happens if and only if $|C(z)| < 1$.

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