On the approximation of real numbers with algebraic integers of low degree

by

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I hereby declare that I am the sole author of this research paper. This is a true copy of the research paper, including any required final revisions, as accepted by my examiners.

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Abstract

Diophantine approximation studies the approximation of real numbers with rational numbers. Diophantine approximation also inspired the mathematical community to study the approximation of real numbers with algebraic integers of some bounded degree over $\mathbb{Q}$. Davenport and Schmidt focused on two cases: when the bounded degree is two (quadratic) and three (cubic). Namely, they studied which $\tau(n)$ will have the following property: for any real number ξ that is not rational and is not an algebraic number of degree at most n, there are infinitely many algebraic integers α of degree n such that

$$|\xi - \alpha| \leq cH(\alpha)^{-\tau(n)},$$

where $c > 0$ is a constant depending on n and ξ and $H(\alpha)$ denotes the maximum of the absolute value of the coefficients of the irreducible polynomial of α over $\mathbb{Z}$. Davenport and Schmidt in [DS69] proved that $\tau(2) = 2$ and $\tau(3) = \gamma^2 = \frac{3+\sqrt{5}}{2}$, where $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden ratio. However they only succeeded in proving that their $\tau(2)$ is the best possible exponent.

No breakthrough happened until 2004 when Damien Roy proved that their $\tau(3)$ is indeed the best possible exponent. This expository work will principally focus on Roy's work ([Roy04], [Roy03]). He identified a certain class of real numbers (which he calls *extremal real numbers*) by careful construction, and then proved that a subset of extremal real numbers has a constant $c' > 0$ such that

$$|\xi - \alpha| \geq c' H(\alpha)^{-\tau(3)}$$

for any algebraic integers α of degree at most 3 over $\mathbb{Q}$.

We conclude with an overview of work done related to finding the optimal value of $\tau(n)$ for $n \geq 4$. Damien Roy in 2008 suggested a possible way to 'generalize' the extremal real numbers to a higher dimension, and computed a possible – but not the best possible – value of $\tau(4)$. Other than this modest progress by Roy, little progress has been made.

Acknowledgements

First, I would like to thank my supervisor, Professor Cameron Stewart, for giving me guidances, and picking me an interesting topic to explore. I thank the readers – my supervisor Professor Cameron Stewart and the second reader Professor David McKinnon – for their useful and insightful suggestions to improve my draft. I also would like to extend my thanks to the University of Waterloo Pure Mathematics Department, for giving me opportunities to do mathematics in one of the best possible environments – a small and cozy department with friendly, intelligent faculty members and graduate students. I also would like to thank my parents for being attentive to my well-being. While they do not understand anything about my work, their unconditional support was essential.

Dedication

This work is dedicated in honour of my parents, and in honour of Professor Carl Pomerance, my undergraduate thesis adviser who retired June of this year. I would also like to dedicate this work in memory of Gordon Elliot Trousdell (1980/04/23 – 2015/01/21), my Grade 11 physics teacher who untimely passed away. I had the honour of being one of his very first students. Both Mr. Trousdell and Professor Pomerance influenced me not only intellectually but also personally – in particular, with their heartfelt, infectious intellectual enthusiasm, open-mindedness and scholastic humility. I sincerely wish that, throughout my intellectual journey, I emulate the qualities they passed on to me during their mentorship.

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[1] It is a well-known fact within the number theory community who know Carl Pomerance that he has a penchant for creating a full-sentence title (see, for instance, [Pom74] and [HP75]). Indeed, my decision to give this section a full-sentence title was deliberate, in his honour. Happy retirement!

Chapter 1

Introduction

Motivated by a paper from 1961 by E. Wirsing [Wir61], H. Davenport and W. Schmidt considered the problem of approximating real numbers by algebraic integers of bounded degree. In other words, Davenport and Schmidt studied how accurately we can approximate a real number by another real number that is a solution to some monic polynomial of bounded degree. Before stating their main result for the degree two case, we need to introduce the following definition first:

Definition 1.1. Let $\alpha \in \mathbb{C}$, and suppose that $f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial of degree n such that $f(\alpha) = 0$. Then α is said to be an *algebraic number* of degree n. If $f(x)$ is a monic irreducible polynomial, then α is called an *algebraic integer* of degree n.

Definition 1.2. Suppose that $P \in \mathbb{Z}[x]$. Then the *height* of P, denoted $H(P)$, is the maximum absolute value of the coefficients of P. Analogously, if α is an algebraic integer such that $Q(\alpha) = 0$, where $Q \in \mathbb{Z}[x]$ is a monic irreducible polynomial, then we define the *height* of α to be $H(\alpha) := H(Q)$.

Proposition 1.3 ([DS69])**.** *Suppose that $\xi \in \mathbb{R}$ is neither rational nor a quadratic irrational. Then there are infinitely many algebraic integers α of degree at most two which satisfy*

$$0 < |\xi - \alpha| \leq cH(\alpha)^{-2},$$

where $c > 0$ is a constant depending on ξ.

In the next chapter, we will present a general sketch of how the proof goes. Then we will explore the cubic case.

In the same paper, Davenport and Schmidt proved the following theorem, which is the main result from [DS69]:

Theorem 1.4 ([DS69, Theorem 1]). *Suppose that ξ is neither rational nor a quadratic irrational. Then there are infinitely many algebraic integers α of degree at most three which satisfy*

$$0 < |\xi - \alpha| \le cH(\alpha)^{-\gamma^2}, \tag{1.1}$$

where γ is the golden ratio, and $c > 0$ is a constant that depends on ξ.

Unlike the quadratic case, Davenport and Schmidt did not prove that their choice of exponent was optimal. Since then, it had been conjectured that the optimal exponent for the cubic case is 3.

Conjecture 1.5. *Suppose that ξ is neither rational nor a quadratic irrational. Then there are infinitely many algebraic integers α of degree at most three which satisfy, for some constant $c = c(\xi)$*

$$0 < |\xi - \alpha| \le cH(\alpha)^{-3}. \tag{1.2}$$

The question of optimality was left open until 2004, where Damien Roy proved in his two-part paper ([Roy04], [Roy03]) that Theorem 1.4 is indeed the best possible result, thereby disproving Conjecture 1.5. We will chiefly explore Roy's two-part paper. In the first part [Roy04], Roy defines a special type of real numbers called *extremal real numbers.*

Definition 1.6. Let $\xi \in \mathbb{R}$, which is neither rational nor quadratic irrational. Suppose that for any real number $X \ge 1$, there exists a constant $c = c(\xi, X)$ such that the inequalities

$$\begin{aligned} |x_0| &\le X \\ |x_0\xi - x_1| &\le cX^{-1/\gamma} \\ |x_0\xi^2 - x_2| &\le cX^{-1/\gamma} \end{aligned} \tag{1.3}$$

have a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$. Then ξ is said to be an *extremal real number.*

Roy proceeds to provide a way to characterize extremal real numbers, prove the properties of extremal real numbers, and then gives a method to construct extremal real numbers. Then he proves theorems pertaining to the approximation of extremal real numbers with a certain category of numbers: rational numbers, quadratic real numbers, and cubic algebraic integers. We state the cubic result here, due to its importance in the context of this work.

Theorem 1.7 ([Roy04, Theorem 1.5]). *Let ξ be an extremal real number. Then there exists a constant $c > 0$ such that for any algebraic integer $\alpha \in \mathbb{C}$ of degree at most 3, we have*

$$|\xi - \alpha| \ge cH(\alpha)^{-\gamma^2 - 1}.$$

However, the above result was not strong enough to prove or disprove whether γ^2 is optimal, though Roy wrote that it shed some doubt on whether Conjecture 1.5 is true. In the second part [Roy03], Roy proved that γ^2 is the optimal choice, thereby disproving the conjecture that 3 is the optimal choice. In order to circumvent the obstacles in the first part, he constructed a subclass of extremal real numbers, and then proved that any extremal real number in this particular subclass of extremal real numbers can 'barely' satisfy the desired measure of approximation stated in Theorem 1.1. The following theorem makes the previous remark more precise:

Theorem 1.8 ([Roy03, Theorem 1.1]). *There exists a real number ξ which is transcendental over $\mathbb{Q}$ and a constant $c > 0$ such that, for any algebraic integer α of degree most 3 over $\mathbb{Q}$, we have*

$$|\xi - \alpha| \geq cH(\alpha)^{-\gamma^2}.$$

Also, the celebrated Schmidt subspace theorem stated below implies that any extremal real number is transcendental (see [Sch80, Chapter VI, Theorem 1B] and [Roy04, p48]). Therefore, it suffices to find an extremal real number satisfying Theorem 1.8. The full rigorous proof of this claim is beyond the scope of this work. We therefore shall state the Schmidt subspace theorem instead.

Theorem 1.9 (Schmidt subspace theorem). *Let $L_1(\mathbf{x}), L_2(\mathbf{x}), \cdots, L_n(\mathbf{x})$ be linearly independent linear forms in n variables whose coefficients consist of algebraic numbers. Then for any real $\varepsilon > 0$, the non-zero integer points $\mathbf{x}$ satisfying*

$$|L_1(\mathbf{x})L_2(\mathbf{x}) \cdots L_n(\mathbf{x})| < |\mathbf{x}|^{-\varepsilon}$$

lie in a finite number of proper subspaces of the vector space $\mathbb{Q}^n$.

Proof. See [Sch72]. □

This paper will be organized in the following manner. Chapter 2 discusses the problem in the quadratic case. While this case is relatively simple, as Davenport and Schmidt stated in [DS69], the argument for the quadratic case is instructive in understanding the cubic case. Chapter 3 focuses on §2–6 in [Roy04]. In §3 and §4 of [Roy04] Roy introduces the necessary tools and notation to characterize extremal real numbers. §5 provides a theorem ([Roy04, Theorem 5.1]) that characterizes extremal real numbers, which is used to prove the additional properties of extremal real numbers ([Roy04, Corollaries 5.2, 5.3, 5.4]). §2 and §6 are devoted to constructing a class of extremal real numbers. This paper is organized slightly differently: after introducing the tools and proving [Roy04, Theorem 5.1], we will focus on the construction, and then prove [Roy04, Corollaries 5.2, 5.3, 5.4]. Chapter 4 focuses on the approximation results related to extremal real numbers (specifically, §7–9 of [Roy04] and the entire [Roy03]). Chapter 5 discusses the quartic case. While little progress has been made for the degree four case, Roy, in [Roy08], attempts to generalize the notion of extremal real numbers, and with those, he comes up with a plausible exponent for the quartic case. He remarked that upon further investigation, he proved that the exponent he came up with is not optimal. Finally, throughout this work, $\gamma = (1 + \sqrt{5})/2$ will be the golden ratio.

Chapter 2

Quadratic case

As Davenport and Schmidt did in their joint paper [DS69], we will briefly explore what happens when we try to approximate a real number with quadratic algebraic integers. As they acknowledged at the beginning of the paper, this case is simple, certainly simpler than what is going to follow after this chapter. Due to the simplicity of the argument, the quadratic case is great for instructive purposes: that is, this case, while being accessible, will illustrate some tools and ideas used in this field, and what exactly "optimal" signifies in this context. For this reason, we shall devote an entire chapter to exploring the quadratic case, "even though this is very simple" (quoting from the same paper [DS69]). We hope that this chapter will help the reader appreciate the more involved cubic case, the case that we will devote most of our energy on.

2.1 Approximation of irrational numbers

First we define the following notation, which will be used throughout this paper:

Definition 2.1.1. We write $A \ll B$ whenever there exists a positive constant c such that $A \leq cB$. Furthermore, if $A \ll B \ll A$, then we write $A \asymp B$.

Let ξ be irrational. We start by noting that there are infinitely many integer pairs $(a, b) \in \mathbb{Z}^2$ such that

$$0 < |\xi^2 + a\xi + b| \ll |a|^{-1} \tag{2.1}$$

by theorems of Minkowski and Hurwitz.

Factoring $x^2 + ax + b = (x - \alpha)(x - \alpha')$ for some quadratic integer α and its conjugate α', we see that

$$\begin{aligned}(\alpha - \alpha')^2 &= (\alpha + \alpha')^2 - 4\alpha\alpha' \\ &= a^2 - 4b = a^2 + 4a\xi + 4\xi^2 - 4a\xi - 4\xi^2 - 4b \\ &= (2\xi + a)^2 + O(1).\end{aligned}$$

Therefore it follows that either $|\xi - \alpha| \gg |\alpha - \alpha'| \gg |2\xi + a| \gg |a|$ or $|\xi - \alpha'| \gg |2\xi + a| \gg |a|$. Suppose, without loss of generality, that

$$|\xi - \alpha'| \gg |2\xi + a| \gg |a|. \tag{2.2}$$

Note that

$$0 < |(\xi - \alpha)(\xi - \alpha')| \ll |a|^{-1} \tag{2.3}$$

by (2.1). Thus by (2.2) and (2.3),

$$0 < |\xi - \alpha| \ll |a|^{-2} \ll H(\alpha)^{-2}.$$

Thus, we proved that

Proposition 2.1.2. *Suppose that ξ is irrational. Then there are infinitely many algebraic integers α of degree at most 2 such that*

$$|\xi - \alpha| \leq cH(\alpha)^{-2}$$

for some constant $c > 0$ depending on ξ.

2.2 Approximation of rational numbers

When ξ is a rational number, our bound is less sharp than the irrational case. Nonetheless, we will tackle this case also for the sake of completeness. Note that this time we can find infinitely many $(a, b) \in \mathbb{Z}^2$ such that

$$0 < |\xi^2 + a\xi + b| \leq 1. \tag{2.4}$$

Factoring $x^2 + ax + b = (x - \alpha)(x - \alpha')$, we see that

$$(\alpha - \alpha')^2 = (2\xi + a)^2 + O(1).$$

Therefore, using the same argument as we did in the irrational case, it follows that either $|\xi - \alpha| \gg |a|$ or $|\xi - \alpha'| \gg |a|$. Suppose, without loss of generality, that $|\xi - \alpha'| \gg |a|$. Since $0 < |(\xi - a)(\xi - a')| \leq 1$, it follows that

$$0 < |\xi - \alpha| \ll |a|^{-1} \ll H(\alpha)^{-1}.$$

This proves the following proposition:

Proposition 2.2.1. *Suppose that ξ is rational. Then there are infinitely many algebraic integers α of degree at most 2 such that*

$$|\xi - \alpha| \leq c' H(\alpha)^{-1}$$

for some constant $c' > 0$ depending on ξ.

2.3 On the optimality of Proposition 2.1.2

Now we discuss if Proposition 2.1.2 is the best possible, or optimal. That is, we need to prove that if the 2 in the exponent is replaced with some other number greater than 2, then there are no longer infinitely many algebraic integers of degree at most 2 over $\mathbb{Q}$ satisfying the measure of approximation. That is, if there exists an irrational number ξ and a constant $c'' > 0$ so that for *any* algebraic integer of degree at most 2 over $\mathbb{Q}$, we have

$$|\xi - \alpha| \geq c'' H(\alpha)^{-2}.$$

We assume that ξ is a quadratic irrational, i.e., an irrational number that is the zero of a quadratic polynomial. Thus ξ has an algebraic conjugate. Call the algebraic conjugate ξ'. Suppose that $(a, b) \in \mathbb{Z}^2$ is chosen so that $|\xi^2 + a\xi + b| \neq 0$. Then we have that

$$|\xi^2 + a\xi + b| \cdot |\xi'^2 + a\xi' + b| = |(\xi^2 + a\xi + b)(\xi'^2 + a\xi' + b)|.$$

Note that $|(\xi^2 + a\xi + b)(\xi'^2 + a\xi' + b)| \gg 1$ by our choice of a and b; so it follows that

$$|\xi^2 + a\xi + b| \gg |\xi'^2 + a\xi' + b|^{-1} \gg |a|^{-1}.$$

Therefore, using the similar argument as we did in the irrational case, we see that indeed $|\xi - \alpha| \gg H(\alpha)^{-2}$. We shall now formally state what we just proved, which also proves that the exponent 2 in Proposition 2.1.2 is the best possible hence cannot be improved further.

Proposition 2.3.1. *Suppose that ξ is a quadratic irrational. Then there exists a constant $c'' > 0$ such that*

$$|\xi - \alpha| \geq c'' H(\alpha)^{-2}$$

for any α, where α is an algebraic integer of degree at most 2 over $\mathbb{Q}$.

Chapter 3

Extremal real numbers

3.1 Key tools and notation

Definition 3.1.1. Let $\mathbf{x} = (x_0, x_1, x_2), \mathbf{y} = (y_0, y_1, y_2), \mathbf{z} = (z_0, z_1, z_2) \in \mathbb{Z}^3$. We define

$$\det(\mathbf{x}) := \det \begin{bmatrix} x_0 & x_1 \\ x_1 & x_2 \end{bmatrix}$$
$$\det(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}.$$

Definition 3.1.2. For any $\xi \in \mathbb{R}$ and $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$, we let

$$\|\mathbf{x}\| := \max\{|x_0|, |x_1|, |x_2|\}$$
$$L(\mathbf{x}) := \max\{|x_0\xi - x_1|, |x_0\xi^2 - x_2|\}.$$

For any square matrix A, we define $\|A\| := |\det(A)|$.

We conclude this section by proving a lemma which provides useful estimates regarding $\|\mathbf{x}\|$ and $L(\mathbf{x})$.

Lemma 3.1.3 ([Roy04, Lemma 3.1]). *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^3$.

(a) For any choice of integers $r, s, t, u \in \{0, 1, 2\}$ *such that* $s - r = u - t$*, we have*

$$\left\| \begin{matrix} x_r & x_s \\ y_t & y_u \end{matrix} \right\| \ll \|\mathbf{x}\| L(\mathbf{y}) + \|\mathbf{y}\| L(\mathbf{x}).$$

(b) The following holds:

$$|\det(\mathbf{x}, \mathbf{y}, \mathbf{z})| \ll \|\mathbf{x}\| L(\mathbf{y}) L(\mathbf{z}) + \|\mathbf{y}\| L(\mathbf{x}) L(\mathbf{z}) + \|\mathbf{z}\| L(\mathbf{x}) L(\mathbf{y}).$$

Proof. This follows from [DS69], in particular from the computations in the proofs of Lemmas 3 and 4, and from the multilinearity of the determinant. □

3.2 The sequence of minimal points

Let ξ be a real number that is neither rational nor quadratic over $\mathbb{Q}$. Consider the finite set

$$S_X := \{\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3 : 1 \le x_0 \le X, L(\mathbf{x}) \le 1\}.$$

We remark that S_X necessarily contains one, and only one, point that minimizes $L(\mathbf{x})$. This follows from the observation that $L(\mathbf{x}) = L(\mathbf{y})$ if and only if $\mathbf{x} = \pm\mathbf{y}$.

Definition 3.2.1. Let $\mathbf{x} \in S_X$ be the integral point that minimizes $L : \mathbb{Z}^3 \to \mathbb{R}$. Then $\mathbf{x}$ is said to be the *minimal point corresponding to* X.

If $\mathbf{x}$ is a minimal point for some real number $X \ge 1$, then it is necessarily a minimal point for x_0. This implies that we can order the points in S_X according to x_0, the first coordinates of such points. Observe that the coordinates of a minimal point must be relatively prime. To see this, suppose otherwise: that is, $\gcd(x_0, x_1, x_2) = d > 1$ where $\mathbf{x} = (x_0, x_1, x_2)$ is a minimal point. Then we have $L(\mathbf{x}) = dL(\mathbf{x}/d)$, which contradicts the minimality of $\mathbf{x}$. Thus it follows that any two distinct minimal points are $\mathbb{Q}$-linearly independent, as vectors in $\mathbb{Q}^3$. Specifically, any two consecutive (with respect to ordering according to x_0) minimal points are $\mathbb{Q}$-linearly independent.

The following lemma gives us an estimation property related to a minimal point. We require introducing a new definition before stating the lemma.

Definition 3.2.2. Let A be a 2×3 matrix and V a two-dimensional subspace of $\mathbb{Q}^3$. Then the *height of* A, denoted $H(A)$, is the maximal absolute value of its minors of order 2. The *height of* V, denoted $H(V)$, is the height of any 2×3 matrix whose rows form a basis of $V\cap\mathbb{Z}^3$.

Lemma 3.2.3. *Let* $\mathbf{x}$ *be a minimal point, and let* $\mathbf{y}$ *be the next minimal point. Let* $V = \langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{Q}}$ *be the two-dimensional subspace (of* $\mathbb{Q}^3$*) generated by* $\mathbf{x}$ *and* $\mathbf{y}$*. Then* $\{\mathbf{x},\mathbf{y}\}$ *is a* $\mathbb{Z}$*-basis of* $V\cap\mathbb{Z}^3$*, and also* $H(V)\asymp \|\mathbf{y}\|L(\mathbf{x})$*.*

Proof. Let $\mathbf{x}=(x_0,x_1,x_2)$ and $\mathbf{y}=(y_0,y_1,y_2)$. Suppose that $\{\mathbf{x},\mathbf{y}\}$ is not a basis of $V\cap\mathbb{Z}^3$. Then there exists $\mathbf{z}=(z_0,z_1,z_2)\in\mathbb{Z}^3$ with $z_0>0$ such that $\mathbf{z}=r\mathbf{x}+s\mathbf{y}$ where $r,s\in\mathbb{Q}$ and $\max\{|r|,|s|\}\le\frac{1}{2}$. Thus the triangle inequality implies

$$z_0\le |r|x_0+|s|y_0<y_0 \text{ and } L(\mathbf{z})\le |r|L(\mathbf{x})+|s|L(\mathbf{y})<L(\mathbf{x}).$$

Hence, $\mathbf{z}$ is the next minimal point after $\mathbf{x}$, which is a contradiction. Hence $\{\mathbf{x},\mathbf{y}\}$ forms a basis of V as desired. If we let $A:=\begin{bmatrix} x_0 & x_1 & x_2\\ y_0 & y_1 & y_2\end{bmatrix}$, then by Lemma 3.1.3(a) gives us

$$H(V)=H(A)\ll \|\mathbf{y}\|L(\mathbf{x})+\|\mathbf{x}\|L(\mathbf{y})\ll\|\mathbf{y}\|L(\mathbf{x}).$$

Thus it remains to show that $H(V)\gg\|\mathbf{y}\|L(\mathbf{x})$. Let $u_i=x_i-x_0\xi^i$ and $v_i=y_i-y_0\xi^i$, where $i=1,2$. Let $j\in\{1,2\}$ such that $|u_j|=L(\mathbf{x})$. It follows that, by reverse-triangle inequality,

$$\begin{aligned} H(V)\ge \left\|\begin{matrix} x_0 & x_j\\ y_0 & y_j\end{matrix}\right\| = |x_0y_j-x_jy_0| &= |x_0y_j-x_0y_0\xi^j+x_0y_0\xi^j-x_jy_0|\\ &=|x_0v_j-y_0u_j|\ge y_0|u_j|-x_0|v_j|\\ &\ge (y_0-x_0)L(\mathbf{x}).\end{aligned}\tag{3.1}$$

Clearly if $\mathbf{z}=\mathbf{y}-\mathbf{x}$, then $\mathbf{z}$ has $z_0:=y_0-x_0$ as its first coordinate with $1\le z_0<y_0$. But since $\mathbf{y}$ is the next minimal point after $\mathbf{x}$, we can choose an index $i\in\{1,2\}$ so that $|z_i-z_0\xi^i|=|v_i-u_i|>L(\mathbf{x})$. This means that we have

$$H(V)\ge|x_0v_i-y_0u_i|\ge|x_0(v_i-u_i)-(y_0-x_0)u_i|\ge x_0L(\mathbf{x})-(y_0-x_0)L(\mathbf{x}).\tag{3.2}$$

Thus (3.2) $\times 2+$ (3.1) gives us

$$3H(V)\ge x_0L(\mathbf{x})+(y_0-x_0)L(\mathbf{x})=y_0L(\mathbf{x})$$

from which the result follows. $\square$

3.3 Characterization of extremal real numbers

Recall the definition of extremal real numbers in Definition 1.6:

Definition 1.6. Let $\xi \in \mathbb{R}$, which is neither rational nor quadratic irrational. Suppose that for any real number $X \geq 1$, there exists a constant $c = c(\xi, X)$ such that the inequalities

$$|x_0| \leq X$$
$$|x_0\xi - x_1| \leq cX^{-1/\gamma}$$
$$|x_0\xi^2 - x_2| \leq cX^{-1/\gamma}$$

have a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$. Then ξ is said to be an *extremal real number.*

The main objective in this section is to prove the following theorem:

Theorem 3.3.1 ([Roy04, Theorem 5.1])**.** *The following are equivalent:*

(a) $\xi \in \mathbb{R}$ *is extremal.*

(b) There exist an increasing sequence of positive integers $(Y_k)_{k\geq 1}$ *and a sequence of points* $(\mathbf{y}_k)_{k\geq 1}$ *of* $\mathbb{Z}^3$ *associated with* ξ *such that, for all* $k \geq 1$*, we have*
  - *(i)* $Y_{k+1} \asymp Y_k^{\gamma}$
  - *(ii)* $\|\mathbf{y}_k\| \asymp Y_k$
  - *(iii)* $L(\mathbf{y}_k) \asymp Y_k^{-1}$
  - *(iv)* $1 \leq |\det(\mathbf{y}_k)| \ll 1$
  - *(v)* $1 \leq |\det(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{y}_{k+2})| \ll 1.$

Before delving into the proof, we introduce a few lemmas from [DS69] which we need in the proof of Theorem 3.3.1.

Lemma 3.3.2. *Suppose that* $(\mathbf{x}_i)_{i\geq 1}$ *is a sequence of minimal points ordered with respect to the first coordinates, and that* $\mathbf{x}_i = (x_{i,0}, x_{i,1}, x_{i,2})$ *for each* i*. Then there exists an index* i_0 *such that for any* $i \geq i_0$*, we have*

$$\left\| \begin{matrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{matrix} \right\| \neq 0.$$

Proof. This is [DS69, Lemma 2]. □

Lemma 3.3.3. *Suppose that* $(\mathbf{x}_i)_{i\geq 1}$ *is a sequence of minimal points ordered with respect to the first coordinates. Then for infinitely many* i*, the points* $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ *are linearly independent over* $\mathbb{Q}$*.*

Proof. This is [DS69, Lemma 5]. □

Proof of Theorem 3.3.1. ((*a*) ⇒ (*b*)) Suppose that ξ is an extremal real number, and let $(\mathbf{x}_i)_{i\geq 1}$ be the sequence of minimal points, ordered with respect to their first coordinates. For the sake of simplicity of notation, let $L_i := L(\mathbf{x}_i)$ and let X_i be the first coordinate of $\mathbf{x}_i$. Then by Lemma 3.3.2, there exists an index i_0 so that $\det(\mathbf{x}_i) \neq 0$ for all $i \geq i_0$. Also, Lemma 3.3.3 shows that there exist infinitely many indices greater than i_0 so that $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent over $\mathbb{Q}$. Let I be the set of integers i satisfying the following properties:

- $\det(\mathbf{x}_j) \neq 0$ for all $j \geq i$; and
- there exist infinitely many $k > i$ such that $\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}$ are $\mathbb{Q}$-linearly independent.

Let i_m be the m-th smallest element of I. Now we are ready to construct a subsequence of $(\mathbf{x}_k)_{k\geq 1}$ that satisfies the properties (i)–(v). Define $(\mathbf{y}_k)_{k\geq 1}$ to be

$$\mathbf{y}_k := \mathbf{x}_{i_k} \text{ and } Y_k := X_{i_k} \ (k \geq 1).$$

For the simplicity of notation, we shall fix an arbitrary m and define $i = i_m$. Putting the definition of minimal points and the definition of extremal real numbers together, we see that there exists some $c > 0$ such that $L_j \leq cX_{j+1}^{-1/\gamma}$ for any $j \geq 1$. Hence we have

$$L_{i-1} \ll X_i^{-1/\gamma} \text{ and } L_i \ll X_{i+1}^{-1/\gamma}. \tag{3.3}$$

Therefore, it follows that

$$\begin{aligned} 1 \leq |\det(\mathbf{x}_i)| &\ll \|\mathbf{x}_i\| L_i + \|\mathbf{x}_i\| L_i \ll X_i L_i \quad (\text{by Lemma 3.1.3}(a)) \\ &\ll X_i X_{i+1}^{-1/\gamma} \quad (\text{by } (3.3)). \end{aligned} \tag{3.4}$$

Hence $1 \ll X_i X_{i+1}^{-1/\gamma}$, or equivalently $X_{i+1} \ll X_i^{\gamma}$. Recall that $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent over $\mathbb{Q}$, so we have

$$\begin{aligned} 1 \leq |\det(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1})| &\ll \|\mathbf{x}_{i-1}\| L_i L_{i+1} + \|\mathbf{x}_i\| L_{i-1} L_{i+1} + \|\mathbf{x}_{i+1}\| L_{i-1} L_i \\ &\ll X_{i+1} L_i L_{i-1} \ll X_{i+1}^{1-1/\gamma} L_{i-1} = X_{i+1}^{1/\gamma^2} L_{i-1} \ (\because \gamma^2 - \gamma - 1 = 0) \\ &\ll X_{i+1}^{1/\gamma^2} X_i^{-1/\gamma} \ll X_i^{1/\gamma} X_i^{-1/\gamma} = 1. \end{aligned}$$

From this, we have

$$X_{i+1} \asymp X_i^{\gamma}, \quad L_{i-1} \asymp X_i^{-1/\gamma} \quad \text{and } L_i \asymp X_{i+1}^{-1/\gamma} \asymp X_i^{-1}. \tag{3.5}$$

Recall that $\mathbf{x}_i = \mathbf{y}_k$ and $X_i = Y_k$, so the estimates obtained in (3.5) translates into

$$\|\mathbf{y}_k\| \asymp Y_k, \quad L(\mathbf{y}_k) \asymp Y_k^{-1} \quad \text{and } 1 \leq |\det(\mathbf{y}_k)| \ll 1. \tag{3.6}$$

This proves that $(\mathbf{y}_k)_{k\geq 1}$ satisfies (ii), (iii), and (iv).

Define $V := \langle \mathbf{x}_i, \mathbf{x}_{i+1}\rangle_{\mathbb{Q}}$, the two-dimensional vector subspace of $\mathbb{Q}^3$ generated by $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$. Suppose that $j \geq i+1$ is the largest integer satisfying $\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_j \in V$ but $\mathbf{x}_{j+1} \notin V$. Recall that any two consecutive minimal points are linearly independent over $\mathbb{Q}$, so in particular it follows that

$$\langle \mathbf{x}_i, \mathbf{x}_{i+1}\rangle_{\mathbb{Q}} = \langle \mathbf{x}_{j-1}, \mathbf{x}_j\rangle_{\mathbb{Q}}. \tag{3.7}$$

Since $\mathbf{x}_{j+1} \notin V$, the points $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$ are $\mathbb{Q}$-linearly independent. We claim that $j = i_{k+1} \in I$. Indeed, since j is the largest integer such that $\mathbf{x}_{j-1}, \mathbf{x}_j \in V$ but $\mathbf{x}_{j+1} \notin V$, j is the smallest integer greater than than i such that $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$ are linearly independent over $\mathbb{Q}$. Therefore (3.5) holds for j also.

Combining (3.5) with Lemma 3.2.3 together, we obtain

$$\begin{aligned} H(\langle \mathbf{x}_i, \mathbf{x}_{i+1}\rangle_{\mathbb{Q}}) \asymp \|\mathbf{x}_{i+1}\| L_i \asymp X_{i+1} L_i \asymp X_i^{1/\gamma}, \text{ and} \\ H(\langle \mathbf{x}_{j-1}, \mathbf{x}_j\rangle_{\mathbb{Q}}) \asymp X_j L_{j-1} \asymp X_j^{1/\gamma^2}. \end{aligned}$$

(3.7) yields therefore $X_j^{1/\gamma^2} \asymp X_i^{1/\gamma}$, or $X_j \asymp X_i^{\gamma}$. Also since $X_i = Y_k$ and $X_j = Y_{k+1}$, we have $Y_{k+1} \asymp Y_k^{\gamma}$.

By the definition of i and j, we see that $\langle \mathbf{y}_k, \mathbf{y}_{k+1}\rangle_{\mathbb{Q}} = \langle \mathbf{x}_i, \mathbf{x}_j\rangle_{\mathbb{Q}}$ contains $\mathbf{x}_{i+1}$ and $\mathbf{x}_{j-1}$. Similarly, $\langle \mathbf{y}_{k+1}, \mathbf{y}_{k+2}\rangle_{\mathbb{Q}}$ contains $\mathbf{x}_{j+1}$. Therefore we have $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1} \in \langle \mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{y}_{k+2}\rangle_{\mathbb{Q}}$. Since $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$ are linearly independent, the points $\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{y}_{k+2}$ are linearly independent over $\mathbb{Q}$ also. We apply Lemma 3.1.3(b) and the estimate $Y_{k+1} \asymp Y_k^{\gamma}$, and we get

$$\begin{aligned} 1 \leq |\det(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{y}_{k+2})| &\ll \|\mathbf{y}_{k+2}\| L(\mathbf{y}_k) L(\mathbf{y}_{k+1}) \\ &\ll Y_{k+2} Y_k^{-1} Y_{k+1}^{-1} \ll Y_{k+2} Y_k^{-1-\gamma} \\ &\ll Y_k^{\gamma^2-\gamma-1} = 1. \end{aligned}$$

Thus $(\mathbf{y}_k)_{k\geq 1}$ and $(Y_k)_{k\geq 1}$ satisfy all of (i)–(v) as required.

($(b) \Rightarrow (a)$) Suppose that $(\mathbf{y}_k)_{k\geq 1}$ and $(Y_k)_{k\geq 1}$ are the sequences for ξ satisfying all the desired properties. First, we claim that ξ cannot be rational or quadratic irrational. Suppose otherwise. That is, there exist $p, q, r \in \mathbb{Z}$ not all zero such that

$$p + q\xi + r\xi^2 = 0. \tag{3.8}$$

Write $\mathbf{y}_k = (y_{k,0}, y_{k,1}, y_{k,2})$; then we have, by (3.8) and the triangle inequality, that

$$\begin{aligned}|py_{k,0} + qy_{k,1} + ry_{k,2}| &= |y_{k,0}(-q\xi - r\xi^2) + qy_{k,1} + ry_{k,2}| \\ &= |q(y_{k,1} - y_{k,0}\xi) + r(y_{k,2} - y_{k,0}\xi^2)| \\ &\leq |q(y_{k,1} - y_{k,0}\xi)| + |r(y_{k,2} - y_{k,0}\xi^2)| \\ &\leq \max\{|q|, |r|\} L(\mathbf{y}_k) \ll L(\mathbf{y}_k) \ll Y_k^{-1}.\end{aligned}$$

Therefore, since (Y_k) is a monotone-increasing sequence, for any sufficiently large k, it follows that $|py_{k,0} + qy_{k,1} + ry_{k,2}| = 0$ since $Y_k^{-1} \to 0$ as $k \to \infty$. Thus we would have $(\mathbf{y}_k) = (0, 0, 0)$ for all sufficiently large k, contradicting property (v). Note that, since $(Y_k)_{n\geq 1}$ is an increasing sequence, for any real number $X \geq 1$ there exists an index k such that $Y_k \leq X < Y_{k+1}$. Per property (ii), we have $\|\mathbf{y}_k\| \asymp Y_k \ll X$. Properties (i) and (iii) give $L(\mathbf{y}_k) \asymp Y_k^{-1} \asymp Y_{k+1}^{-1/\gamma} \ll X^{-1/\gamma}$. Hence $\mathbf{y}_k$ is an element in $\mathbb{Z}^3$ satisfying (1.3). The claim follows. □

3.4 Construction of extremal real numbers

Now that we have an alternate way to characterize extremal real numbers – namely in terms of a sequence of elements in $\mathbb{Z}^3$ – we are ready to construct extremal real numbers using $\mathbb{Z}^3$. First, we will identify $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ with the symmetric matrix

$$\mathbf{x} = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_2 \end{bmatrix}.$$

Thus, from now on we shall use $\mathbf{x}$ in two different senses, either as an element in $\mathbb{Z}^3$ or as its associated symmetric matrix. For

$$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we make the following observations, where $\mathbf{x} = (x_0, x_1, x_2), \mathbf{y} = (y_0, y_1, y_2), \mathbf{z} = (z_0, z_1, z_2)$.

First, we have

$$\begin{aligned}\mathrm{tr}(J\mathbf{x}J\mathbf{z}J\mathbf{y}) &= \mathrm{tr}\left(\begin{bmatrix} x_1 & x_2 \\ -x_0 & -x_1 \end{bmatrix}\begin{bmatrix} z_1 & z_2 \\ -z_0 & -z_1 \end{bmatrix}\begin{bmatrix} y_1 & y_2 \\ -y_0 & -y_1 \end{bmatrix}\right) \\ &= x_0\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} - x_1\begin{vmatrix} y_0 & y_2 \\ z_0 & z_2 \end{vmatrix} + x_2\begin{vmatrix} y_0 & y_1 \\ z_0 & z_1 \end{vmatrix} = \det(\mathbf{x}, \mathbf{y}, \mathbf{z}).\end{aligned} \tag{3.9}$$

Second, we see that

$$-\mathbf{x}J\mathbf{z}J\mathbf{y} = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} z_2 & -z_1 \\ -z_1 & z_0 \end{bmatrix} \begin{bmatrix} y_0 & y_1 \\ y_1 & y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} x_0 & x_1 \\ \begin{vmatrix} y_0 & y_1 \\ z_0 & z_1 \end{vmatrix} & \begin{vmatrix} y_0 & y_1 \\ z_1 & z_2 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} x_0 & x_1 \\ \begin{vmatrix} y_1 & y_2 \\ z_0 & z_1 \end{vmatrix} & \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} x_1 & x_2 \\ \begin{vmatrix} y_0 & y_1 \\ z_0 & z_1 \end{vmatrix} & \begin{vmatrix} y_0 & y_1 \\ z_1 & z_2 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} x_1 & x_2 \\ \begin{vmatrix} y_1 & y_2 \\ z_0 & z_1 \end{vmatrix} & \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \end{vmatrix} \end{bmatrix},$$

and that

$$\begin{vmatrix} x_0 & x_1 \\ \begin{vmatrix} y_1 & y_2 \\ z_0 & z_1 \end{vmatrix} & \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \end{vmatrix} - \begin{vmatrix} x_1 & x_2 \\ \begin{vmatrix} y_0 & y_1 \\ z_0 & z_1 \end{vmatrix} & \begin{vmatrix} y_0 & y_1 \\ z_1 & z_2 \end{vmatrix} \end{vmatrix} = \det(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

Thus $-\mathbf{x}J\mathbf{z}J\mathbf{y}$ is symmetric if and only if $\det(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$.

Definition 3.4.1. We shall define $[\mathbf{x}, \mathbf{y}, \mathbf{z}] := -\mathbf{x}J\mathbf{z}J\mathbf{y}$.

Lemma 3.4.2. *Let* $\mathbf{w} = [\mathbf{x}, \mathbf{x}, \mathbf{y}]$. *Then*

$$\|\mathbf{w}\| \ll \|\mathbf{x}\|^2 L(\mathbf{y}) + \|\mathbf{y}\| L(\mathbf{x})^2 \text{ and } L(\mathbf{w}) \ll (\|\mathbf{x}\| L(\mathbf{y}) + \|\mathbf{y}\| L(\mathbf{x})) L(\mathbf{x}).$$

Proof. If $\mathbf{w} = (w_0, w_1, w_2)$, then since $\mathbf{w} = [\mathbf{x}, \mathbf{x}, \mathbf{y}]$ we have

$$w_0 = \begin{vmatrix} x_0 & x_1 \\ \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} & \begin{vmatrix} x_0 & x_1 \\ y_1 & y_2 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} x_0 & x_1 \\ \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} & \begin{vmatrix} x_0 & x_1 \\ y_1 - y_0\xi & y_2 - y_1\xi \end{vmatrix} \end{vmatrix}$$

From the definition of $L(\mathbf{x})$ it follows $L(\mathbf{x}) \ll \|\mathbf{x}\|$. Also, by Lemma 3.1.3(a), we have

$$\|\mathbf{w}\| \ll |w_0| \ll \|\mathbf{x}\| \left\| \begin{matrix} x_0 & x_1 \\ y_1 - y_0\xi & y_2 - y_1\xi \end{matrix} \right\| + L(\mathbf{x}) \left\| \begin{matrix} x_0 & x_1 \\ y_0 & y_1 \end{matrix} \right\| \ll \|\mathbf{x}\|^2 L(\mathbf{y}) + \|\mathbf{y}\| L(\mathbf{x})^2.$$

For the second part, for any $j = 0, 1$, the following computation shows that, again by Lemma 3.1.3(a),

$$|w_{j+1} - w_j\xi| = \left\| \begin{matrix} x_1 - x_0\xi & x_2 - x_1\xi \\ \begin{vmatrix} x_j & x_{j+1} \\ y_0 & y_1 \end{vmatrix} & \begin{vmatrix} x_j & x_{j+1} \\ y_1 & y_2 \end{vmatrix} \end{matrix} \right\| \ll L(\mathbf{x})(\|\mathbf{x}\| L(\mathbf{y}) + \|\mathbf{y}\| L(\mathbf{x})),$$

so we are done. □

Lemma 3.4.3 ([Roy04, Lemma 2.1])**.** *Suppose* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^3$ *such that* $\det(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$*. Let* $\mathbf{w}$ *be an arbitrary element in* $\mathbb{Z}^3$*. Then the following hold:*

(a) $\det[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \det(\mathbf{x}) \det(\mathbf{y}) \det(\mathbf{z})$*;*
(b) $\det(\mathbf{w}, \mathbf{y}, [\mathbf{x}, \mathbf{y}, \mathbf{z}]) = \det(\mathbf{y}) \det(\mathbf{w}, \mathbf{z}, \mathbf{x})$*;*
(c) $\det(\mathbf{x}, \mathbf{y}, [\mathbf{x}, \mathbf{y}, \mathbf{z}]) = 0$*;*
(d) $[\mathbf{x}, \mathbf{y}, [\mathbf{x}, \mathbf{y}, \mathbf{z}]] = \det(\mathbf{x}) \det(\mathbf{y})\mathbf{z}$*.*

Proof. Part (a) immediately follows from the fact that

$$\det(-\mathbf{x}J\mathbf{z}J\mathbf{y}) = (-1)^2 \det(\mathbf{x}) \det(J) \det(\mathbf{z}) \det(J) \det(\mathbf{y}) = \det(\mathbf{x}) \det(\mathbf{y}) \det(\mathbf{z}),$$

since $\det(J) = 1$. For the remaining parts, we first observe that, for any $\mathbf{w} \in \mathbb{Z}^3$, we have

$$\mathbf{w}J\mathbf{w}J = J\mathbf{w}J\mathbf{w} = -\det(\mathbf{w})I, \tag{3.10}$$

where I is the identity matrix. From this, part (d) immediately follows:

$$-\mathbf{x}J(-\mathbf{x}J\mathbf{z}J\mathbf{y})J\mathbf{y} = (\mathbf{x}J\mathbf{x}J)\mathbf{z}(J\mathbf{y}J\mathbf{y}) = \det(\mathbf{x}) \det(\mathbf{y})\mathbf{z}.$$

Part (b) and (c) follow from (3.9) and (3.10):

$$\begin{aligned} \det(\mathbf{w}, \mathbf{y}, [\mathbf{x}, \mathbf{y}, \mathbf{z}]) &= \operatorname{tr}(J\mathbf{w}J(-\mathbf{x}J\mathbf{z}J\mathbf{y})J\mathbf{y}) \\ &= \operatorname{tr}(-J\mathbf{w}J\mathbf{x}J\mathbf{z}(J\mathbf{y}J\mathbf{y})) \\ &= \operatorname{tr}(J\mathbf{w}J\mathbf{x}J\mathbf{z} \det(\mathbf{y})I) \\ &= \det(\mathbf{y}) \operatorname{tr}(J\mathbf{w}J\mathbf{x}J\mathbf{z}) \\ &= \det(\mathbf{y}) \det(\mathbf{w}, \mathbf{z}, \mathbf{x}). \end{aligned}$$

Hence, $\det(\mathbf{x}, \mathbf{y}, [\mathbf{x}, \mathbf{y}, \mathbf{z}]) = \det(\mathbf{y}) \det(\mathbf{x}, \mathbf{z}, \mathbf{x}) = \det(\mathbf{y}) \cdot 0 = 0$, as desired. □

We need to prove one additional lemma before proving the main result:

Lemma 3.4.4 ([Roy04, Lemma 6.1])**.** *Suppose that* A *and* B *are non-commuting symmetric matrices in* $\mathrm{GL}_2(\mathbb{Z})$*. Suppose also that* $(\mathbf{y}_k)_{k \geq -1}$ *is the sequence such that*

$$\begin{gathered} \mathbf{y}_{-1} = B^{-1}, \quad \mathbf{y}_0 = I, \quad \mathbf{y}_1 = A, \\ \mathbf{y}_k = [\mathbf{y}_{k-1}, \mathbf{y}_{k-1}, \mathbf{y}_{k-3}] \text{ for } k \geq 2, \end{gathered} \tag{3.11}$$

where I *is the identity matrix. Then we have, for any* $k \geq -1$,

$$|\det(\mathbf{y}_k)| = 1 \text{ and } |\det(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{y}_{k+2})| = |\operatorname{tr}(JAB)| \neq 0. \tag{3.12}$$

Finally, for any $k \geq 1$ the following recursive relation holds:

$$\mathbf{y}_k = \pm \mathbf{y}_{k-1} S \mathbf{y}_{k-2} \text{ where } S = \begin{cases} AB & (k \equiv 1 \pmod 2) \\ BA & (k \equiv 0 \pmod 2). \end{cases} \tag{3.13}$$

Proof. For any $k \geq 2$, the recurrence relation as defined in (3.11) and Lemma 3.4.3(a), (b) imply that

$$\det(\mathbf{y}_k) = \det(\mathbf{y}_{k-1})^2 \det(\mathbf{y}_{k-3}) \tag{3.14}$$

$$\begin{aligned} \det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k) &= \det(\mathbf{y}_{k-1}) \det(\mathbf{y}_{k-2}, \mathbf{y}_{k-3}, \mathbf{y}_{k-1}) \\ &= -\det(\mathbf{y}_{k-1}) \det(\mathbf{y}_{k-3}, \mathbf{y}_{k-2}, \mathbf{y}_{k-1}). \end{aligned} \tag{3.15}$$

Since $\det(A) = \det(B) = 1$, it follows that $\det(\mathbf{y}_k) = \pm 1$ for $k = -1, 0, 1$. Thus (3.14) implies that $|\det(\mathbf{y}_k)| = 1$ and $|\det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k)| = |\det(\mathbf{y}_{k-3}, \mathbf{y}_{k-2}, \mathbf{y}_{k-1})|$. By recursion it follows $|\det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k)| = |\det(\mathbf{y}_{-1}, \mathbf{y}_0, \mathbf{y}_1)|$. Furthermore, we deduce from the identity $\operatorname{tr}(J\mathbf{x}J\mathbf{z}J\mathbf{y}) = \det(\mathbf{x}, \mathbf{y}, \mathbf{z})$ that

$$|\det(\mathbf{y}_{-1}, \mathbf{y}_0, \mathbf{y}_1)| = |\operatorname{tr}(JBJA)| = |\det(B)\operatorname{tr}(JAB)| = |\operatorname{tr}(JAB)| \neq 0,$$

since A and B do not commute. Finally, suppose that the recurrence (3.13) holds for some $k \geq 1$. Then

$$\mathbf{y}_{k+1} = [\mathbf{y}_k, \mathbf{y}_k, \mathbf{y}_{k-2}] = \pm \mathbf{y}_{k-1} S (\mathbf{y}_{k-2} J \mathbf{y}_{k-2} J) \mathbf{y}_k = \pm \det(\mathbf{y}_{k-2}) \mathbf{y}_{k-1} S \mathbf{y}_k = \pm \mathbf{y}_{k-1} S \mathbf{y}_k.$$

Taking the transpose gives $\mathbf{y}_{k+1} = \pm \mathbf{y}_k (S^t) \mathbf{y}_{k-1}$. The final part of the lemma follows upon verifying that the relation holds when $k = 1$. □

Now we are ready to state the main result, which gives one possible method to construct extremal real numbers:

Theorem 3.4.5 ([Roy04, Theorem 6.2])**.** *Let A, B and $(\mathbf{y}_k)_{k \geq -1}$ be as in Lemma 3.4.4. Assume that all entries of A are non-negative and all entries of AB are positive. For $\mathbf{y}_k = (y_{k,0}, y_{k,1}, y_{k,2})$ and for any $k \geq 2$, we have $y_{k,0} \in \mathbb{Z} \setminus \{0\}$. Furthermore, the limit*

$$\xi := \lim_{k \to \infty} \frac{y_{k,1}}{y_{k,0}}$$

exists, and ξ is an extremal real number. Let $Y_k = \|\mathbf{y}_k\|$ for all k. Then the sequences $(\mathbf{y}_k)_{k \geq 3}$ and $(Y_k)_{k \geq 3}$ satisfy (i)–(v) in Theorem 3.3.1.

Proof. By Lemma 3.4.4, we may assume (3.12). We claim that $\mathbf{y}_k$ has non-zero entries and that all three of them have the same sign. Indeed, note that $\mathbf{y}_0$ and $\mathbf{y}_1$ have non-zero entries and that AB and its transpose $(AB)^T = B^T A^T = BA$ have positive entries; therefore, the recurrence relation (3.13) implies that $\mathbf{y}_{k-1} S \mathbf{y}_{k-2}$ must consist of positive entries, as desired. Thus necessarily $Y_k = \|\mathbf{y}_k\|$ is positive for any $k \geq 2$. The recurrence relation (3.13) also implies that, for any $k \geq 4$, there exists a constant $c_1 > 1$ so that

$$Y_{k-2}Y_{k-1} < Y_k \leq c_1 Y_{k-2} Y_{k-1}. \tag{3.16}$$

Since Y_k is positive for all $k \geq 4$, it follows that the sequence $(Y_k)_{k\geq 3}$ is unbounded and monotone-increasing; that is, $Y_k \to \infty$ as $k \to \infty$. One can apply a similar argument on $y_{k,0}$ to obtain $y_{k,0} \to \infty$ as $k \to \infty$.

Note that (3.13) shows that there exist r, s, t, u depending on k so that

$$\begin{bmatrix} y_{k,0} & y_{k,1} \\ y_{k,1} & y_{k,2} \end{bmatrix} = \pm \begin{bmatrix} y_{k-1,0} & y_{k-1,1} \\ y_{k-1,1} & y_{k-1,2} \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = S\mathbf{x}_{k-2}.$$

Let

$$I_k := \left[\frac{y_{k,1}}{y_{k,0}}, \frac{y_{k,2}}{y_{k,1}}\right]$$

for all $k \geq 2$. We observe that $\mathbb{R} \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Indeed, note that

$$\frac{y_{k,1}}{y_{k,0}} = \frac{ry_{k-1,1} + ty_{k-1,2}}{ry_{k-1,0} + ty_{k-1,1}}, \frac{y_{k,2}}{y_{k,1}} = \frac{sy_{k-1,1} + uy_{k-1,2}}{sy_{k-1,0} + uy_{k-1,1}} \in I_{k-1}.$$

Additionally, since $|I_k| = 1/(y_{k,0}y_{k,1})$ and $y_{k,0} \to \infty$ as $k \to \infty$, it follows that, as $k \to \infty$ we have $|I_k| \to 0$. In other words, for some positive $\xi \in \mathbb{R}$, we have

$$\bigcap_{k=2}^{\infty} I_k = \{\xi\}.$$

Therefore

$$\xi = \lim_{k\to\infty} \frac{y_{k,1}}{y_{k,0}}$$

as desired. Hence $|y_{k,0}| \asymp |y_{k,1}| \asymp |y_{k,2}|$, and since $Y_k = \|\mathbf{y}_k\|$, it follows

$$|y_{k,0}| \asymp |y_{k,1}| \asymp |y_{k,2}| \asymp Y_k.$$

Thus $|I_k| \asymp Y_k^{-2}$, so consequently

$$L(\mathbf{y}_k) \asymp \max_{l=0,1} |y_{k,l}\xi - y_{k,l+1}| \asymp Y_k \max_{l=0,1} \left| \xi - \frac{y_{k,l+1}}{y_{k,l}} \right| \asymp Y_k |I_k| \asymp Y_k^{-1}.$$

So far, we proved that $(\mathbf{y}_k)_{k\geq 3}$ and $(Y_k)_{k\geq 3}$ satisfy properties (ii) and (iii) in Theorem 3.3.1. As for property (i), we observe that, by (3.16), we conclude that by letting $q_k := Y_k Y_{k-1}^{-\gamma}$,

$$\begin{aligned} Y_{k-2}Y_{k-1}^{1-\gamma} &< Y_k Y_{k-1}^{-\gamma} \leq c_1 Y_{k-2} Y_{k-1}^{1-\gamma} \\ Y_{k-2}Y_{k-1}^{-1/\gamma} &< q_k \leq c_1 Y_{k-2} Y_{k-1}^{-1/\gamma} \quad (\because \gamma^2 - \gamma - 1 = 0) \\ q_{k-1}^{-1/\gamma} &< q_k \leq c_1 q_{k-1}^{-1/\gamma} \end{aligned}$$

for any $k \geq 4$. Therefore, for all $k \geq 3$, by letting $c_2 := \max\{c_1^\gamma, q_3, q_3^{-\gamma}\}$ we have $c_2^{-1/\gamma} \leq q_k \leq c_2$, as (q_k) is a decreasing sequence. Hence $Y_{k-1}^\gamma \ll Y_k \ll Y_{k-1}^\gamma$. Finally, properties (iv) and (v) immediately follow, since the sequences were chosen so that (3.12) is satisfied. That ξ is extremal follows upon applying Theorem 3.3.1. □

We introduce a particular kind of extremal real number. To do so, we define Fibonacci words first.

Definition 3.4.6. The *Fibonacci word* on $\{a, b\}$ is a word constructed by using the following rules:

- $S_0 := a$ and $S_1 := ab$.
- $S_n = S_{n-1}S_{n-2}$ (concatenation).

Definition 3.4.7. Define the continued fraction

$$\xi_{a,b} := [0, a, b, a, a, b, \dots] = 1/(a + 1/(b + \cdots)),$$

where the portion $a, b, a, a, b, \dots$ denotes the Fibonacci word on $\{a, b\}$ of infinite length.

The following proposition follows as a corollary to Theorem 3.4.5.

Proposition 3.4.8 ([Roy04, Corollary 6.3])**.** *Let a, b be distinct positive integers. Then $\xi_{a,b}$ is an extremal real number. In particular, it is the special case of the construction using Theorem 3.4.5 with*

$$A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b & 1 \\ 1 & 0 \end{bmatrix}.$$

3.5 Properties of extremal real numbers

We prove some properties of extremal real numbers that will be useful in proving approximation properties of extremal real numbers. Unless otherwise specified, we shall denote $\mathcal{E}$ the set of extremal real numbers.

Corollary 3.5.1. *$\mathcal{E}$ is infinite.*

Proof. This follows from the fact that $\xi_{a,b} \in \mathcal{E}$ for any two distinct positive integers a, b and that the numbers $\xi_{a,b}$ are distinct for different pairs (a, b). $\square$

Proposition 3.5.2 ([Roy04, Corollary 5.2])**.** *Let ξ be an extremal real number and let $(\mathbf{y}_k)_{k\geq 1}$ and $(Y_k)_{k\geq 1}$ be as in the statement of Theorem 3.3.1. Then the point $\mathbf{y}_{k+1}$ is a non-zero rational multiple of $[\mathbf{y}_k, \mathbf{y}_k, \mathbf{y}_{k-2}]$.*

Proof. Suppose $k \geq 4$, and write $\mathbf{w} = [\mathbf{y}_k, \mathbf{y}_k, \mathbf{y}_{k+1}]$. By Lemma 3.4.3(a), it follows that

$$\det(\mathbf{w}) = \det(\mathbf{y}_k)^2 \det(\mathbf{y}_{k+1}).$$

Therefore $\det(\mathbf{w}) \neq 0$ so $\mathbf{w} \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$. Then Lemma 3.4.2 and properties (i)–(iii) from Theorem 3.3.1 imply that

$$\begin{aligned} \|\mathbf{w}\| &\ll Y_k^2 Y_{k+1}^{-1} \asymp Y_{k-2}, \text{ and} \\ L(\mathbf{w}) &\ll Y_{k+1} Y_k^{-2} \asymp Y_{k-2}^{-1}. \end{aligned}$$

We deduce from Lemma 3.1.3(b) that

$$\begin{aligned} |\det(\mathbf{w}, \mathbf{y}_{k-3}, \mathbf{y}_{k-2})| &\ll \|\mathbf{w}\| L(\mathbf{y}_{k-2}) L(\mathbf{y}_{k-3}) \asymp Y_{k-2} Y_{k-2}^{-1} Y_{k-3}^{-1} \asymp Y_{k-3}^{-1} \\ |\det(\mathbf{w}, \mathbf{y}_{k-2}, \mathbf{y}_{k-1})| &\ll Y_{k-1} Y_{k-2}^{-2} \asymp Y_{k-3}^{-1/\gamma}. \end{aligned}$$

Since the left-hand side of both of the above inequalities must be an integer, for sufficiently large k, the left-hand side must be zero. But since the three consecutive points in $(\mathbf{y}_k)_{k\geq 1}$ are linearly independent over $\mathbb{Q}$, $\mathbf{w}$ must be a rational multiple of $\mathbf{y}_{k-2}$. Finally, since

$$[\mathbf{y}_k, \mathbf{y}_k, \mathbf{w}] = \det(\mathbf{y}_k)^2 \mathbf{y}_{k+1},$$

the claim follows. $\square$

The following consequence of Proposition 3.5.2 will be especially useful in Section 4.3.2:

Proposition 3.5.3 ([Roy03, Proposition 2.3]). *Suppose ξ is an extremal real number and $(\mathbf{x}_k)_{k\geq 1}$ be the sequence in Theorem 3.3.1. Then there exists an integer $k_0 \geq 1$ and $M \in \mathrm{M}_2(\mathbb{Z})$ such that $\mathbf{x}_{k+2}$ (as a symmetric matrix, as explained in Section 3.4) is a rational multiple of*

- $\mathbf{x}_{k+1} M \mathbf{x}_k$ *for all $k \geq k_0$ odd, and*
- $\mathbf{x}_{k+1}(M^t)\mathbf{x}_k$ *for all $k \geq k_0$ even.*

Proof. By Proposition 3.5.2, indeed there exists an integer $k_0 \geq 1$ so that $\mathbf{x}_{k+2}$ is a rational multiple of $[\mathbf{x}_{k+1}, \mathbf{x}_{k+1}, \mathbf{x}_{k-1}] = \mathbf{x}_{k+1}\mathbf{x}_{k-1}^{-1}\mathbf{x}_{k+1}$ for all $k > k_0$. Pick some $k > k_0$ such that $\mathbf{x}_{k+1}$ is a rational multiple of $\mathbf{x}_k S \mathbf{x}_{k-1}$ for some suitable S, say $\mathbf{x}_{k+1} = g\mathbf{x}_k S \mathbf{x}_{k-1}$ for some $g \in \mathbb{Q}$. Then we have, for a suitable $h \in \mathbb{Q}$, that

$$\mathbf{x}_{k+2} = h\mathbf{x}_{k+1} x_{k-1}^{-1} \mathbf{x}_{k+1} = h(g\mathbf{x}_k S \mathbf{x}_{k-1})\mathbf{x}_{k-1}^{-1}\mathbf{x}_{k+1} = (hg)\mathbf{x}_k S \mathbf{x}_{k+1}.$$

Hence $\mathbf{x}_{k+2}$ is a rational multiple of $\mathbf{x}_k S \mathbf{x}_{k+1}$. Therefore once we show that the base case holds, then the claim will follow by induction. But the base case ($k = k_0$) is immediate, upon choosing the right matrix M so that the desired property holds. □

Remark 3.5.4. Observe that when all the $\mathbf{x}_k$'s have determinant 1, we may assume that $M \in \mathrm{GL}_2(\mathbb{Z})$. Then we have, where $S = M$ or M^t depending on the parity of k,

$$\begin{aligned} \det(\mathbf{x}_k) &= \det(g\mathbf{x}_{k-1} S \mathbf{x}_{k-2}) = g^2 \det(\mathbf{x}_{k-1}) \det(S) \det(\mathbf{x}_{k-2}) \\ 1 &= g^2. \end{aligned}$$

Therefore in this case $g = \pm 1$, where S is either M or M^t depending on the parity of k.

Proposition 3.5.5 ([Roy04, Corollary 5.3]). *In the notation of Theorem 3.5.2, the following identity is true, provided that k is sufficiently large:*

$$\det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k)\mathbf{y}_{k+1} = \det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_{k+1})\mathbf{y}_k + \det(\mathbf{y}_{k-1}, \mathbf{y}_k, \mathbf{y}_{k+1})\mathbf{y}_{k-2}.$$

Proof. Lemma 3.1.3(b) implies that $\det(\mathbf{y}_{k-2}, \mathbf{y}_k, [\mathbf{y}_k, \mathbf{y}_k, \mathbf{y}_{k-2}]) = 0$. Therefore, from this and Proposition 3.5.2 we conclude that $\mathbf{y}_{k-2}, \mathbf{y}_k, \mathbf{y}_{k+1}$ are linearly dependent over $\mathbb{Q}$. Thus there exist $a, b \in \mathbb{Q}$ not all zero such that $\mathbf{y}_{k+1} = a\mathbf{y}_k + b\mathbf{y}_{k-2}$. Therefore, by the multilinearity of the determinant,

$$\begin{aligned} \det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k)\mathbf{y}_{k+1} &= a\det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k)\mathbf{y}_k + b\det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k)\mathbf{y}_{k-2} \\ &= \det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_{k+1})\mathbf{y}_k + \det(\mathbf{y}_{k+1}, \mathbf{y}_{k-1}, \mathbf{y}_k)\mathbf{y}_{k-2}, \end{aligned}$$

since

$$\det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}) = a \det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k)$$

and

$$\det(\mathbf{y}_{k-1}, \mathbf{y}_k, \mathbf{y}_{k+1}) = b \det(\mathbf{y}_{k-1}, \mathbf{y}_k, \mathbf{y}_{k-2}).$$ □

Proposition 3.5.6 ([Roy04, Corollary 5.4])**.** *$\mathcal{E}$ is at most countable.*

Proof. Consider the following map $\phi : \mathcal{E} \to (\mathbb{Z}^3)^3$ defined as follows. For each extremal real number $\xi \in \mathcal{E}$, choose a corresponding sequence $(\mathbf{y}_k)_{k\geq 1}$ using Theorem 3.3.1. Now choose a sufficiently large index $i \geq 3$ so that whenever $k \geq i$, the point $\mathbf{y}_{k+1}$ is a rational multiple of $[\mathbf{y}_k, \mathbf{y}_k, \mathbf{y}_{k-2}]$, per Proposition 3.5.2. Define $\phi(\xi) := (\mathbf{y}_{i-2}, \mathbf{y}_{i-1}, \mathbf{y}_i)$. Again by Proposition 3.5.2, once we know $\mathbf{y}_{i-2}, \mathbf{y}_{i-1}, \mathbf{y}_i$, we can construct $\mathbf{y}_k$ for all $k \geq i-2$ up to a non-zero rational multiple. Therefore the ratio $y_{k,1}/y_{k,0}$ is uniquely determined. We deduce that $\xi = \lim_{k\to\infty} y_{k,1}/y_{k,0}$ is uniquely determined also. This proves that ϕ is injective, as required. □

Therefore, combining with Corollary 3.5.1, we deduce that $\mathcal{E}$ is countably infinite. Also recall the discussion on page 3, where we stated that an extremal real number is transcendental over $\mathbb{Q}$. Hence, we have the following theorem.

Theorem 3.5.7 ([Roy04, Theorem 1.1])**.** *Let ξ be an extremal real number. Then ξ is transcendental over $\mathbb{Q}$ and $\mathcal{E}$ is countable.*

Chapter 4

Approximation of extremal real numbers

4.1 Approximation by rational numbers

We prove a key result on the approximation property of extremal real numbers with rational numbers. The main result, Theorem 4.1.3, follows as a consequence of the proposition stated below. The proof of the proposition involves ideas from geometry of numbers, prompting us to introduce the following definition:

Definition 4.1.1. Suppose that $\mathcal{C}$ is a convex body. Let λ be a number such that the boundary of $\lambda\mathcal{C}$ contains a lattice point (a point whose coordinates are all integers), but not the interior of $\lambda\mathcal{C}$. Then λ is the *first minimum of* $\mathcal{C}$.

Proposition 4.1.2 ([Roy04, Proposition 7.1])**.** *Let ξ be an extremal real number. Then there exists a positive constant s so that the first minimum of the convex body*

$$\mathcal{C}(X) := \{(x_0, x_1) \in \mathbb{R}^2 : |x_0| \leq X \text{ and } |x_0\xi - x_1| \leq X^{-1}\}$$

is bounded below by $(\log X)^{-s}$, for any sufficiently large real number X.

Proof. Suppose $(\mathbf{y}_k)_{k\geq 1}$ and $(Y_k)_{k\geq 1}$ are the sequences associated with ξ (i.e., satisfies the properties listed in Theorem 3.3.1). Suppose that X is a real number with $X \geq Y_2$, and $\lambda := \lambda(X)$ the first minimum of $\mathcal{C}(X)$. Suppose that (x_0, x_1) is the point of $\mathbb{Z}^2$ which realizes this minimum. In other words, we have $|x_0| \le \lambda X$ and $|x_0\xi - x_1| \le \lambda X^{-1}$. Choose an index k so that $Y_k \le X \le Y_{k+1}$. Define $(z_0, z_1) \in \mathbb{Z}^2$ as follows:

$$(z_0, z_1) = (x_0, x_1)J\mathbf{y}_{k+1}J\mathbf{y}_{k-1},$$

following the convention listed at the beginning of Section 3.4. Write $\mathbf{y}_{k-1} = (y_0^*, y_1^*, y_2^*)$ and $\mathbf{y}_{k+1} = (y_0', y_1', y_2')$. We deduce that

$$\begin{aligned}
|x_0| &= \left\| \begin{matrix} x_0 & x_1 \\ \begin{vmatrix} y_0^* & y_1^* \\ y_0' & y_1' \end{vmatrix} & \begin{vmatrix} y_0^* & y_1^* \\ y_1' & y_2' \end{vmatrix} \end{matrix} \right\| \\
&= \left\| \begin{matrix} x_0 & x_0\xi - x_1 \\ \begin{vmatrix} y_0^* & y_1^* \\ y_0' & y_1' \end{vmatrix} & \begin{vmatrix} y_0^* & y_1^* \\ y_0'\xi - y_1' & y_1'\xi - y_2' \end{vmatrix} \end{matrix} \right\| \\
&\ll |x_0|Y_k^{-1} + |x_0\xi - x_1|Y_k \quad (\because \text{Lemma 3.1.3(a) and Theorem 3.3.1}) \\
&\ll \lambda X Y_k^{-1}. \quad (\because Y_k \le X)
\end{aligned}$$

and

$$\begin{aligned}
|x_0\xi - x_1| &= \left\| \begin{matrix} x_0 & x_1 \\ \begin{vmatrix} y_0^*\xi - y_1^* & y_1^*\xi - y_2 \\ y_0' & y_1' \end{vmatrix} & \begin{vmatrix} y_0^*\xi - y_1^* & y_1^*\xi - y_2^* \\ y_1' & y_2' \end{vmatrix} \end{matrix} \right\| \\
&= \left\| \begin{matrix} x_0 & x_0\xi - x_1 \\ \begin{vmatrix} y_0^*\xi - y_1^* & y_1^*\xi - y_2 \\ y_0' & y_1' \end{vmatrix} & \begin{vmatrix} y_0^*\xi - y_1^* & y_1^*\xi - y_2^* \\ y_0'\xi - y_1' & y_0\xi' - y_2' \end{vmatrix} \end{matrix} \right\| \\
&\ll |x_0|Y_{k-1}^{-1}Y_{k+1}^{-1} + |x_0\xi - x_1|Y_k \quad (\because \text{Lemma 3.1.3(a) and Theorem 3.3.1}) \\
&\ll \lambda Y_k X^{-1}.
\end{aligned}$$

Therefore, if we let $Z = 2XY_k^{-1}$ then the two relations above imply that the first minimum $\lambda(Z)$ of $\mathcal{C}(Z)$ satisfies $\lambda(Z) \le c\lambda(X)$ for some constant $c > 0$ that does not depend on k and X. Note also that $Z \ll X^{1/\gamma^2}$, hence $Z \le X^{1/2}$ for any sufficiently large $X \ge X_0 \ge Y_2$. Choose s large enough so that $2^{s-1} \ge c$. Then it follows that, for any $X \ge X_0$ (hence $2 \le Z \le X^{1/2}$),

$$\lambda(X)(\log X)^s \ge 2\lambda(Z)(\log Z)^s.$$

Hence we see that $\lambda(X)(\log X)^s$ is bounded below by a positive constant for $2 \le X \le X_0^{1/2}$, as required. Observe that, therefore, $\lambda(X)(\log X)^s$ tends to infinity as $X \to \infty$. $\square$

Suppose ξ is extremal and $\alpha = p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$ and $q \geq 3$. Furthermore, suppose that $|\xi - \alpha| \leq q^{-2}(\log q)^{-2s-2}$. Define X and λ as stated in Proposition 4.1.2, and in particular define $X = q(\log q)^{s+1}$ and $\lambda = (\log q)^{-s-1}$. Following the same line of argument as in the above proof, we see that $|q| \leq \lambda X$ and $|q\xi - p| \leq \lambda X^{-1}$. Therefore the first minimum of $\mathcal{C}(X)$ is at most λ. We deduce that $\lambda \geq (\log X)^{-s}$ provided that q (hence X) is sufficiently large, per Proposition 4.1.2. Due to the way we defined X and λ, we see that the inequality $\lambda \geq (\log X)^{-s}$ gives us an upper bound for q. Therefore the value of $H(\alpha)$ has an upper bound also. Thus, Theorem 4.1.3 follows upon letting $t = 2s + 2$.

Theorem 4.1.3 ([Roy04, Theorem 1.3]). *Let ξ be an extremal real number. Then there exists $c, t > 0$ so that for any rational number $\alpha \in \mathbb{Q}$ we have*

$$|\xi - \alpha| \geq cH(\alpha)^{-2}(1 + \log H(\alpha))^{-t}.$$

4.2 Approximation by quadratic algebraic numbers

In this section we tackle the approximation properties of extremal real numbers with algebraic numbers of degree at most 2. The main results are Theorems 4.2.4 and 4.2.5.

Proposition 4.2.1 ([Roy04, Proposition 8.1]). *Let ξ be an extremal real number and let the sequence $(Y_k)_{k\geq 1}$ and $(\mathbf{y}_k)_{k\geq 1}$ satisfy properties (i)–(v) in Theorem 3.3.1. For each $k \geq 1$, define*

$$Q_k(T) := \begin{vmatrix} 1 & T & T^2 \\ y_{k,0} & y_{k,1} & y_{k,2} \\ y_{k+1,0} & y_{k+1,1} & y_{k+1,2} \end{vmatrix}.$$

Then Q_k is a polynomial with integral entries which, for all sufficiently large values of k, has degree 2 and satisfies

$$H(Q_k) \asymp |Q_k'(\xi)| \asymp Y_{k-1} \text{ and } |Q_k(\xi)| \asymp Y_{k+2}^{-1}.$$

Proof. We begin by observing that applying Lemma 3.1.3(a) gives us

$$H(Q_k) = \left\| \begin{matrix} y_{k,0} & y_{k,1} & y_{k,2} \\ y_{k+1,0} & y_{k+1,1} & y_{k+1,2} \end{matrix} \right\| \ll Y_{k+1}L(\mathbf{y}_k) + Y_kL(\mathbf{y}_{k+1}) \asymp Y_{k-1}^{\gamma^2-\gamma} = Y_{k-1}.$$

Suppose that k is sufficiently large throughout the remaining portion of this proof. Note that

$$Q_k'' = 2\begin{vmatrix} y_{k,0} & y_{k,1} - y_{k,0}\xi \\ y_{k+1,0} & y_{k+1,1} - y_{k+1,0}\xi \end{vmatrix} = 2y_{k+1,0}(y_{k,0}\xi - y_{k,1}) + O(Y_{k-1}^{-1}),$$

we see that $Q_k'' \neq 0$, as the lower bound in Theorem 4.1.3 implies that $|y_{k,0}\xi - y_{k,1}|$ is positive. A similar type of argument gives us

$$Q_k'(\xi) = -\begin{vmatrix} y_{k,0} & y_{k,2} - 2y_{k,1}\xi + y_{k,0}\xi^2 \\ y_{k+1,0} & y_{k+1,2} - 2y_{k+1,1}\xi + y_{k+1,0}\xi^2 \end{vmatrix}. \tag{4.1}$$

Thus for any $j \geq 1$, we have

$$y_{j,0}(y_{j,2} - 2y_{j,1}\xi + y_{j,0}\xi^2) = \det(\mathbf{y}_j) + (y_{j,1} - y_{j,0}\xi)^2 = \det(\mathbf{y}_j) + O(Y_j^{-2}),$$

so equivalently

$$|y_{j,2} - 2y_{j,1}\xi + y_{j,0}\xi^2| \asymp Y_j^{-1}.$$

The above relation allows us to estimate (4.1); in particular,

$$|Q_k'(\xi)| = |y_{k+1,0}||y_{k,2} - 2y_{k,1}\xi + y_{k,0}\xi^2| + O(Y_{k-1}^{-1}) \asymp Y_{k-1},$$

and since $H(Q_k) \ll Y_{k-1}$, we get $H(Q_k) \asymp Y_{k-1}$. Define a new vector $\mathbf{z}$ as follows:

$$\mathbf{z} := y_{k+2,0}(1, \xi, \xi^2) - \mathbf{y}_{k+2}.$$

Therefore, we have $\|\mathbf{z}\| \ll Y_{k+2}^{-1}$. Multilinearity of the determinant gives

$$y_{k+2,0}Q_k(\xi) = \det(\mathbf{y}_{k+2}, \mathbf{y}_k, \mathbf{y}_{k+1}) + \det(\mathbf{z}, \mathbf{y}_k, \mathbf{y}_{k+1}),$$

and since

$$|\det(\mathbf{z}, \mathbf{y}_k, \mathbf{y}_{k+1})| \ll 3\|\mathbf{z}\|H(Q_k) \ll Y_{k+1}Y_{k+2}^{-1},$$

it follows that

$$|Q_k(\xi)| = |y_{k+2,0}|^{-1}(|\det(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{y}_{k+2})| + O(Y_{k-1}Y_{k+2}^{-1})) \asymp Y_{k+2}^{-1},$$

as required. □

In the proof of the next theorem, we need the following variant of Liouville's inequality:

Lemma 4.2.2. *Suppose that $\alpha, \beta \in \mathbb{C}$ are algebraic numbers. Suppose that $d(\alpha)$ is the degree of the irreducible polynomial of α over $\mathbb{Z}$. Furthermore, for any $P \in \mathbb{Z}[x]$, define*

$$t(P) := \log H(P) + (\log 2) \deg P,$$

and that $t(\alpha) := t(Q)$ where Q is the irreducible polynomial of α over $\mathbb{Z}$. Then we have

$$|\alpha - \beta| \geq 2e^{-d(\alpha)t(\beta) - d(\beta)t(\alpha)}.$$

Proof. This is [LR99, Lemma 3]. □

Theorem 4.2.3 ([Roy04, Theorem 8.2])**.** *Let ξ be an extremal real number. Then there exist an integer $k_0 \geq 1$ and positive constants c_1 and c_2 such that, for any $k \geq k_0$, the polynomial Q_k as in Proposition 4.2.1 is quadratic, is irreducible over $\mathbb{Q}$, and admits* exactly one *root α_k satisfying*

$$c_1 H(\alpha_k)^{-2\gamma^2} \leq |\xi - \alpha_k| \leq c_2 H(\alpha_k)^{-2\gamma^2}. \tag{4.2}$$

There also exists a constant $c_3 > 0$ such that, for any algebraic number $\alpha \in \mathbb{C}$ of degree at most two over $\mathbb{Q}$ except for the algebraic numbers α_k with $k \geq k_0$, we have

$$|\xi - \alpha| \geq c_3 H(\alpha)^{-4}.$$

Proof. Let k be sufficiently large. Then we may assume that, by Proposition 4.2.1, the polynomial $Q_k(T)$ has degree 2. Whenever $Q_k(T)$ has degree 2, we may factor it as $Q_k(T) = a_k(T - \alpha_k)(T - \beta_k)$. Without loss of generality, we may assume that $|\xi - \alpha_k| \leq |\xi - \beta_k|$. Taking the logarithmic derivative gives us

$$\frac{|Q_k'(\xi)|}{|Q_k(\xi)|} = \left|\frac{(\xi - \alpha_k) + (\xi - \beta_k)}{(\xi - \alpha_k)(\xi - \beta_k)}\right| = \left|\frac{1}{\xi - \alpha_k} + \frac{1}{\xi - \beta_k}\right|,$$

from which we have, by the triangle inequality,

$$\frac{1}{|\xi - \alpha_k|} - \frac{1}{|\xi - \beta_k|} \leq \frac{|Q_k'(\xi)|}{|Q_k(\xi)|} \leq \left|\frac{1}{\xi - \alpha_k}\right| + \left|\frac{1}{\xi - \beta_k}\right| \leq \frac{2}{|\xi - \alpha_k|}. \tag{4.3}$$

From (4.3) it follows

$$|\xi - \alpha_k| \leq 2\frac{|Q_k(\xi)|}{|Q_k'(\xi)|};$$

now apply Proposition 4.2.1, and we find that

$$|\xi - \alpha_k| \leq 2\frac{|Q_k(\xi)|}{|Q_k'(\xi)|} \asymp \frac{Y_{k+2}^{-1}}{Y_{k-1}} \asymp Y_{k-1}^{-1-\gamma^3} \asymp H(Q_k)^{-1-\gamma^3} = H(Q_k)^{-2\gamma^2} \ll H(\alpha_k)^{-2\gamma^2}. \tag{4.4}$$

Note that $H(Q_k) \to \infty$ as $k \to \infty$. Therefore $|\xi - \alpha_k| \to 0$ as $k \to \infty$. That is, if k is sufficiently large then α_k must be irrational since ξ is. Thus Q_k is irreducible over $\mathbb{Q}$ as long as k is sufficiently large. Since $\operatorname{disc}(Q_k(T)) = a_k^2(\alpha_k + \beta_k)^2 - 4a_k^2\alpha_k\beta_k = a_k^2(\alpha_k - \beta_k)^2$ and the discriminant must be a positive integer, we have

$$|\alpha_k - \beta_k| \geq |a_k|^{-1} \geq H(Q_k)^{-1}.$$

Furthermore, the observation that $|\alpha_k - \beta_k| \geq 3|\xi - \alpha_k|$ leads us to conclude that, per (4.3),

$$\frac{1}{2|\xi - \alpha_k|} \leq \frac{Q_k'(\xi)}{Q_k(\xi)} \leq \frac{2}{|\xi - \alpha_k|}.$$

Also, the greatest common divisor of the coefficients of $Q_k(T)$ must divide $\det(\mathbf{y}_{k-1}, \mathbf{y}_k, \mathbf{y}_{k+1})$ (recall that $\det(\mathbf{y}_{k-1}, \mathbf{y}_k, \mathbf{y}_{k+1}$ is a non-zero integer per the $\mathbb{Q}$-linear independence of the the three vectors). Recall also that, by property (v) from Theorem 3.3.1, the integer $\det(\mathbf{y}_{k-1}, \mathbf{y}_k, \mathbf{y}_{k+1}$ is bounded above by a constant independent on k. Therefore $H(\alpha_k) \asymp H(Q_k)$ for all sufficiently large k, so the $\ll$ in (4.4) can be replaced with $\asymp$. That is, we can choose two constants c_1, c_2 so that (4.2) holds for all $k \geq k_0 \geq 2$.

For the second part, suppose that $\alpha \in \mathbb{C} \setminus A$ is an algebraic number of degree at most 2, where

$$A := \{\alpha_k : i \geq k_0\}.$$

Then Lemma 4.2.2 gives us

$$\begin{aligned}
|\alpha - \alpha_k| &\geq 2e^{-d(\alpha)t(\alpha_k) - d(\alpha_k)t(\alpha)} \\
&\geq 2e^{-2(\log H(\alpha_k) + 2\log 2) - 2(\log H(\alpha) + 2\log 2)} \\
&= 2e^{-2\log H(\alpha_k)} e^{-4\log 2} e^{-2\log H(\alpha)} e^{-4\log 2} \\
&= c_3 H(\alpha)^{-2} H(\alpha_k)^{-2}
\end{aligned}$$

for some absolute constant c_3. Now let k be the smallest integer with $k \geq k_0$ satisfying

$$H(\alpha) \leq \sqrt{\frac{c_3}{2c_2}} H(\alpha_k)^{\gamma}.$$

Note that such choice of k gives us that

$$\begin{aligned}
&\sqrt{\frac{c_3}{2c_2}} H(\alpha_k)^{\gamma} \geq H(\alpha) \\
\Leftrightarrow \quad & H(\alpha_k)^{2\gamma} \geq \frac{2c_2}{c_3} H(\alpha)^2 \\
\Leftrightarrow \quad & H(\alpha_k)^{-2\gamma} \leq \frac{c_3}{2c_2} H(\alpha)^{-2} \\
\Leftrightarrow \quad & c_2 H(\alpha_k)^{-2\gamma} \leq \frac{c_3}{2} H(\alpha)^{-2} \\
\Leftrightarrow \quad & c_2 H(\alpha_k)^{-2(\gamma+1)} = c_2 H(\alpha_k)^{-2\gamma^2} \leq \frac{c_3}{2} H(\alpha)^{-2} H(\alpha_k)^{-2}.
\end{aligned}$$

Therefore, it follows that

$$|\xi - \alpha_k| \le c_2 H(\alpha_k)^{-2\gamma^2} \le \frac{c_3}{2} H(\alpha)^{-2} H(\alpha_k)^{-2} \le \frac{1}{2}|\alpha - \alpha_k|,$$

and hence

$$|\xi - \alpha| \ge \frac{1}{2}|\alpha - \alpha_k| \gg H(\alpha)^{-2} H(\alpha_k)^{-2}.$$

Since we have $H(\alpha) \gg H(\alpha_k)$ for any $k \ge k_0$, we indeed have $|\xi - \alpha| \ge c_3 H(\alpha)^{-4}$, as required. Indeed, note that if $k > k_0$ then $H(\alpha) \gg H(\alpha_{k-1})^{\gamma} \gg H(\alpha_k)$ while $H(\alpha) \ge 1 \gg H(\alpha_k)$ if $k = k_0$. This completes the proof. □

The following theorem, one of the two main results in this section, is a direct consequence of Proposition 4.2.1.

Theorem 4.2.4 ([Roy04, Theorem 1.2]). *Let ξ be an extremal real number. Then there exists a constant $c > 0$ depending only on ξ such that the inequalities*

$$\begin{aligned} |x_0| &\le X \\ |x_1| &\le X \\ |x_0\xi^2 + x_1\xi + x_2| &\le cX^{-\gamma^2} \end{aligned} \tag{4.5}$$

have a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for any real number $X \ge 1$.

Proof. Suppose $X \ge H(Q_1)$. Then for any X, we can choose an index k so that $H(Q_k) \le X \le H(Q_{k+1})$. Write $Q_k(T) = x_0T^2 + x_1 + x_2$. Then $|x_0|, |x_1| \le X$ since $H(Q_k) \le X$. Finally, by Proposition 4.2.1 and the inequality $X \le H(Q_{k+1})$, we find that

$$|x_0\xi^2 + x_1\xi + x_2| = |Q_k(\xi)| \asymp Y_{k+2}^{-1} \asymp Y_k^{-\gamma^2} \asymp H(Q_{k+1})^{-\gamma^2} \ll X^{-\gamma^2}. \qquad \square$$

Let c_1, c_2, c_3 be the constants as they appear in Theorem 4.2.3. Then the following main result follows from Theorem 4.2.3 upon letting $C := \max\{c_1, c_2\}$ and $C' := c_3$.

Theorem 4.2.5 ([Roy04, Theorem 1.4]). *Let ξ be an extremal real number. Then there exists $C > 0$ such that the inequality*

$$|\xi - \alpha| \le CH(\alpha)^{-2\gamma^2}$$

has infinitely many solutions in quadratic algebraic numbers $\alpha \in \mathbb{C}$. Also, there exists $C' > 0$ so that the inequality

$$|\xi - \alpha| \le C'H(\alpha)^{-2\gamma^2}$$

has no solution in algebraic numbers $\alpha \in \mathbb{C}$ of degree at most two.

4.3 Approximation by cubic algebraic integers

4.3.1 Initial results by Roy

We prove the two propositions Roy proved in [Roy04, §9], which he also uses in proving Theorem 1.8, thereby proving that Theorem 1.4 is the best possible result. For any real number x, we will define $\{x\}$ to be the distance from x to its nearest integer. That is, $\{x\} := \min\{|x - z| : z \in \mathbb{Z}\}$. Finally, throughout this section, we let ξ to be some fixed extremal real number, and $(Y_k)_{k\geq 1}$ and $(\mathbf{y}_k)_{k\geq 1}$ the sequences associated with ξ that satisfy the five properties listed in Theorem 3.3.1. Therefore, it is true that

$$\{y_{k,0}\xi^m\} \ll Y_k^{-1} \tag{4.6}$$

for $m = 0, 1, 2$. We also remark that Theorem 1.7 immediately follows from either one of the two propositions.

Proposition 4.3.1 ([Roy04, Proposition 9.1]). *Suppose that there are δ and c_1 with $0 \leq \delta < 1$ and $c_1 > 0$ so that*

$$\{y_{k,0}\xi^3\} \geq c_1 Y_k^{-\delta} \tag{4.7}$$

for all $k \geq 1$. Then there exists a constant $c_2 > 0$ such that, for any algebraic integer $\alpha \in \mathbb{C}$ of degree at most three, we have

$$|\xi - \alpha| \geq c_2 H(\alpha)^{-\theta},$$

where $\theta = (\gamma^2 + \delta/\gamma)/(1 - \delta)$.

Proof. Let α be an arbitrary algebraic integer of degree at most three. Also suppose that $P(T) := T^3 + pT^2 + qT + r$ be the product of the irreducible polynomial over $\mathbb{Z}$ and an appropriate number of copies of T so that $P(T)$ necessarily has degree three. By definition, $H(\alpha) = H(P)$ so we find that, for all $k \geq 1$,

$$\begin{aligned}\{y_{k,0}\xi^3\} = \{y_{k,0}(P(\xi) - p\xi^2 - q\xi - r)\} &\leq \{y_{k,0}P(\xi)\} + |p|\{y_{k,0}\xi^2\} + |q|\{y_{k,0}\xi\} + |r|\{y_{k,0}\} \\ &\leq \{y_{k,0}P(\xi)\} + (|p| + |q|)Y_k^{-1} \\ &\leq \{y_{k,0}P(\xi)\} + 2H(\alpha)Y_k^{-1} \\ &\leq c_3(Y_k H(\alpha)|\xi - \alpha| + Y_k^{-1}H(\alpha)),\end{aligned}$$

for an appropriate constant $c_3 > 0$, which depends on ξ. Let k be the smallest index satisfying the relation

$$H(\alpha) \leq \frac{c_1}{2c_3} Y_k^{1-\delta}.$$

Now apply (4.7) to see that

$$|\xi - \alpha| \geq \frac{c_1}{2c_3} Y_k^{-1-\delta} H(\alpha)^{-1}.$$

The claim follows upon noting that our choice of k satisfies $Y_k \ll H(\alpha)^{\gamma/(1-\delta)}$. □

We need the following slightly weaker result as well, in Section 4.3.2.

Proposition 4.3.2 ([Roy04, Proposition 9.2])**.** *In the same notation as in Proposition 4.3.1, we have* $\{y_{k,0}\xi^3\} \gg Y_k^{-1/\gamma^3}$ *for all* $k \geq k_0$*, where* k_0 *is sufficiently large so that* $y_{k,0} \neq 0$ *for all* $k \geq k_0$*.*

Proof. Let $k \geq 4$. For some fixed k, let $y_{k,3}$ be the nearest integer from $y_{k,2}\xi$. Write

$$\begin{aligned} \tilde{\mathbf{y}}_k &= (y_{k,1}, y_{k,2}, y_{k,3}) \\ d_{k-2} &= \det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k), \\ \tilde{d}_{k-2} &= \det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \tilde{\mathbf{y}}_k). \end{aligned}$$

But by property (v) in Theorem 3.3.1, we see that d_{k-2} is in a finite set of non-zero integers. Note that ξ is irrational and $\tilde{d}_{k-2}$ is an integer, so by the linearity of determinant, we have

$$1 \ll |d_{k-2}\xi - \tilde{d}_{k-2}|.$$

Meanwhile, Proposition 4.2.1 gives us that

$$\begin{aligned} |d_{k-2}\xi - \tilde{d}_{k-2}| &= |\det(\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \xi\mathbf{y}_k - \tilde{\mathbf{y}}_k)| \\ &\ll \|\xi\mathbf{y}_k - \tilde{\mathbf{y}}_k\| H(Q_{k-2}) \\ &\ll (\{y_{k,2}\xi\} + Y_k^{-1}) Y_{k-3}. \end{aligned}$$

Thus the last line and the fact that $Y_k \asymp Y_{k+1}^{\gamma}$ implies that $\{y_{k,2}\xi\} \gg Y_{k-3}^{-1} \asymp Y_k^{-1/\gamma^3}$ for all sufficiently large k. The proposition follows upon recognizing that $|\xi(y_{k,0}\xi^2 - y_{k,2}| \ll L(\mathbf{y}_k) \ll Y_k^{-1}$. □

4.3.2 The γ^2 in Theorem 1.4 is the best possible result[1]

The chief objective of this section is to prove Theorem 1.8, by constructing a class of extremal real numbers satisfying the desired measure of approximation.

[1] It is a well-known fact within the number theory community who know Carl Pomerance that he has a penchant for creating a full-sentence title (see, for instance, [Pom74] and [HP75]). Indeed, my decision to give this section a full-sentence title was deliberate, in his honour. Happy retirement!

Theorem 4.3.3 ([Roy03, Proposition 2.1]). *The following are equivalent:*

(a) A real number ξ is extremal.
(b) There exists a constant $c_1 \geq 1$ and an unbounded sequence of non-zero points $(\mathbf{x}_k)_{k\geq 1}$ of $\mathbb{Z}^3$ satisfying, for all $k \geq 1$,
  (i) $c_1^{-1}\|\mathbf{x}_k\|^{\gamma} \leq \|\mathbf{x}_{k+1}\| \leq c_1\|\mathbf{x}_k\|^{\gamma}$,
  (ii) $c_1^{-1}\|\mathbf{x}_k\|^{-1} \leq L_{\xi}(\mathbf{x}_k) \leq c_1\|\mathbf{x}_k\|^{-1}$,
  (iii) $1 \leq |\det(\mathbf{x}_k)| \leq c_1$
  (iv) $1 \leq |\det(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{x}_{k+2})| \leq c_1$.

Proof. This is the equivalent formulation of Theorem 3.3.1. □

Proposition 4.3.4 ([Roy03, Proposition 2.2]). *Let ξ be an extremal real number and let $(\mathbf{x}_k)_{k\geq 1}$ be the sequence mentioned in Theorem 4.3.3. Suppose $\mathbf{x}_k = (x_{k,0}, x_{k,1}, x_{k,2})$, and that there exists a constant $c_2 > 0$ so that*

$$\{x_{k,0}\xi^3\} \geq c_2$$

for all $k \geq 1$. Then, for any algebraic integer α of degree 3 or less over $\mathbb{Q}$, there exists a constant $c_3 > 0$ such that

$$|\xi - \alpha| \geq c_3 H(\alpha)^{-\gamma^2}.$$

Proof. This is a special case of Proposition 4.3.1. □

Recall that ξ is transcendental over $\mathbb{Q}$, by Theorem 3.5.7. Therefore it suffices to find extremal real numbers satisfying the assumptions stated in Proposition 4.3.4. Proposition 4.3.2 implies that there exists $c_4 > 0$ satisfying, for any sufficiently large k,

$$\{x_{k,0}\xi^3\} \geq c_4\|\mathbf{x}_k\|^{-1/\gamma^3}.$$

Definition 4.3.5. Let $M \in \mathrm{GL}_2(\mathbb{Z})$ be a non-symmetric matrix. We define $\mathcal{E}(M)$ to be the set of extremal real numbers satisfying the following property. There exists a sequence of points $(\mathbf{x}_k)_{k\geq 1}$ such that

- the conditions listed in Theorem 4.3.3 hold, and
- whenever we view $\mathbf{x}_k$ as symmetric matrices, then $\mathbf{x}_k \in \mathrm{GL}_2(\mathbb{Z})$ and satisfies the relation

$$\mathbf{x}_{k+2} = \mathbf{x}_{k+1}S\mathbf{x}_k \text{ where } S = \begin{cases} M & (k \equiv 1 \pmod 2) \\ M^t & (k \equiv 0 \pmod 2). \end{cases}$$

Now that we introduced $\mathcal{E}(M)$, we are ready to introduce the following lemma:

Lemma 4.3.6 ([Roy03, Lemma 2.5])**.** *Let $\xi \in \mathcal{E}(M)$ where $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is a non-symmetric matrix of determinant 1, and let $(\mathbf{x}_k = (x_{k,0}, x_{k,1}, x_{k,2}))_{k\geq 1}$ be a sequence with the desired properties, as listed in Definition 4.3.5. Then for all $k \geq 2$, the following hold:*

(a) $\mathbf{x}_{k+2} = (ax_{k,0} + (b+c)x_{k,1} + dx_{k,2})\mathbf{x}_{k+1} \pm \mathbf{x}_{k-1}$,
(b) $x_{k,0}x_{k+1,2} - x_{k,2}x_{k+1,0} = \pm(ax_{k-1,0} - dx_{k-1,2}) \pm (b-c)x_{k-1,1}$.

Proof. Both follow from linear algebra. Recall that $\mathbf{x}_{k+1} = \mathbf{x}_k S \mathbf{x}_{k-1}$ and $\mathbf{x}_{k+2} = \mathbf{x}_{k+1}(S^t)\mathbf{x}_k$ where S is M or M^t based on the parity of k. Hence

$$\mathbf{x}_{k+2} = \mathbf{x}_{k+2}^t = \mathbf{x}_k S \mathbf{x}_{k+1} = (\mathbf{x}_k S)^2 \mathbf{x}_{k-1}.$$

The Cayley-Hamilton theorem implies that

$$(\mathbf{x}_k S)^2 = \operatorname{tr}(\mathbf{x}_k S)\mathbf{x}_k S - \det(\mathbf{x}_k S) I,$$

so indeed

$$\mathbf{x}_{k+2} = \operatorname{tr}(\mathbf{x}_k S)\mathbf{x}_{k+1} - \det(\mathbf{x}_k S)\mathbf{x}_{k-1}.$$

This completes the proof of (a). As for (b), notice that the left-hand side is equal to the sum of the entries that are not in the diagonal of the product $\mathbf{x}_k J \mathbf{x}_k^{k+1}$. Furthermore, since $J\mathbf{x}_k J = \pm \mathbf{x}_k^{-1}$, it follows that

$$\mathbf{x}_k J \mathbf{x}_{k+1} = \pm J \mathbf{x}_k^{-1} \mathbf{x}_{k+1} = \pm J \mathbf{x}_k^{-1}(\mathbf{x}_k S \mathbf{x}_{k-1}) = \pm J S \mathbf{x}_{k-1},$$

which is equal to the right-hand side of (b), as required. □

Corollary 4.3.7. *If $(a,b,c,d) = (1,1,-1,0)$, then*

(a) $\mathbf{x}_{k+2} = x_{k,0}\mathbf{x}_{k+1} \pm \mathbf{x}_{k-1}$,
(b) $x_{k,0}x_{k+1,2} - x_{k,2}x_{k+1,0} = \pm x_{k-1,0} \pm 2x_{k-1,1}$.

We now identify a class of extremal real numbers we need:

Theorem 4.3.8. *Any real number ξ in the set*

$$\mathcal{E}_1 := \mathcal{E}\left(\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\right)$$

satisfies the measure of approximation of Theorem 1.4.

Remark 4.3.9. The version in [Roy03] states that the claim holds even when the 1 in the first row and the first column is replaced with an arbitrary positive integer, say a. Roy actually proves only that $\mathcal{E}_1$ is non-empty; this motivates us to prove our result when $a = 1$, especially since Theorem 1.4 requires us to find *a real number*. As for proving the non-emptiness for general a (i.e., $\mathcal{E}_a := \mathcal{E}(\left[\begin{smallmatrix} a & 1 \\ -1 & 0 \end{smallmatrix}\right])$), he remarks that one should define the sequence $(\mathbf{x}_k)_{k\geq 1}$ using the recurrence relation as stated in Definition 4.3.5 with

$$\mathbf{x}_1 := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{x}_2 := \begin{bmatrix} a^3 + 2a & a^3 - a^2 + 2a - 1 \\ a^3 - a^2 + 2a - 1 & a^3 - 2a^2 + 3a - 2 \end{bmatrix}$$

and

$$M := \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix}.$$

Then he writes that one can apply the arguments laid out in [Roy04, §6] (or, equivalently, Section 3.4 in this work) to show that any $\mathbf{x}_k \in \mathrm{GL}_2(\mathbb{Z})$ for all k and that $(\mathbf{x}_k)_{k\geq 1}$ satisfies all the desired properties in Theorem 4.3.3 for some ξ.

First, we need to ensure that $\mathcal{E}_1$ is non-empty.

Proposition 4.3.10. *Let $m \in \mathbb{Z}_+$. Then the real number*

$$\lambda := (m + 1 + \xi_{m,m+2})^{-1} = [0, m+1, m, m+2, m, m, m+2, \dots]$$

is in $\mathcal{E}_1$.

Proof. Suppose that $\xi \in \mathcal{E}(M)$ for some non-symmetric matrix $M \in \mathrm{GL}_2(\mathbb{Z})$; furthermore, let $(\mathbf{x}_k)_{k\geq 1}$ be the sequence of symmetric matrices associated with M. Pick $C \in \mathrm{GL}_2(\mathbb{Z})$ such that $(\lambda, -1) \in \mathbb{R}^2$ is a multiple of $(\xi, -1)C$. Then $\lambda \in \mathcal{E}(C^tMC)$ with the corresponding sequence $(C^{-1}\mathbf{x}_k(C^{-1})^t)_{k\geq 1}$.

Now let

$$M = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m+2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (m+1)^2 & m \\ m+2 & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & -1 \\ -1 & m+1 \end{bmatrix}$$

so that

$$C^tMC = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Indeed, $\xi_{m,m+2} \in \mathcal{E}(M)$ by Proposition 3.4.8. Therefore $\lambda \in \mathcal{E}(C^tMC) = \mathcal{E}_1$. $\square$

Proof of Theorem 4.3.8. Let $\xi \in \mathcal{E}_1$, and suppose that $(\mathbf{x}_k)_{k\geq 1}$ of $\mathbb{Z}^3$ is a corresponding sequence of points associated with ξ, per Definition 4.3.5. For simplicity of notation, define

$$\delta_k := \{x_{k,2}\xi\} \text{ and } X_k := \|\mathbf{x}_k\|,$$

where $\{q\}$ denotes the distance from q to its closest integer. Let c_1 be the constant as in Theorem 4.3.3. Observe that

$$\begin{aligned} &\{x_{k,0}\xi\} \leq |x_{k,0}\xi - x_{k,1}| \leq c_1 X_k^{-1} \\ &\{x_{k,1}\xi\} \leq |x_{k,1}\xi - x_{k,2}| \leq |\xi| \cdot |x_{k,1} - x_{k,0}\xi| + |x_{k,0}\xi^2 - x_{k,2}| \leq (|\xi| + 1)c_1 X_k^{-1}. \end{aligned} \tag{4.8}$$

Corollary 4.3.7(a) gives

$$x_{k+2,2} = x_{k,0}x_{k+1,2} \pm x_{k-1,2}; \tag{4.9}$$

similarly, by Corollary 4.3.7(b), we have

$$x_{k,0}x_{k+1,2} = x_{k,2}x_{k+1,0} \pm x_{k-1,0} \pm 2x_{k-1,1}. \tag{4.10}$$

Thus from (4.8) and and (4.10), we estimate that

$$\begin{aligned} \{x_{k,0}x_{k+1,2}\xi\} &= \{(x_{k,2}x_{k+1,0} \pm x_{k-1,0} \pm 2x_{k-1,1})\xi\} \\ &\leq |x_{k,2}|\{x_{k+1,0}\xi\} + \{x_{k-1,0}\xi\} + 2\{x_{k-1,1}\xi\} \\ &\leq X_k\{x_{k+1,0}\xi\} + \{x_{k-1,0}\xi\} + 2\{x_{k-1,1}\xi\} \\ &\ll X_k X_{k+1}^{-1}. \end{aligned}$$

By applying Theorem 4.3.3(i), we further estimate $X_k X_{k+1}^{-1}$. Since $X_k \leq c_1 X_{k-1}^{\gamma}$ and $X_{k+1}^{-1} \leq c_1 X_k^{-\gamma} \leq c_1(c_1 X_{k-1}^{\gamma})^{-\gamma}$, we deduce

$$X_k X_{k+1}^{-1} \leq c_1 X_{k-1}^{\gamma}(c_1^{1-\gamma} X_{k-1}^{-\gamma^2}) = c_1^{2-\gamma} X_{k-1}^{-(\gamma^2-\gamma)} = c_1^{2-\gamma} X_{k-1}^{-1}.$$

Putting these together, we see that there exists a constant $c_5 > 0$ such that

$$\{x_{k,0}x_{k+1,2}\xi\} \leq c_5 X_{k-1}^{-1}. \tag{4.11}$$

Now it follows from (4.9) and (4.11) that

$$\begin{aligned} |\delta_{k+2} - \delta_{k-1}| &= |\{(x_{k,0}x_{k+1,2} \pm x_{k-1,2})\xi\} - \{x_{k-1,2}\xi\}| \\ &\leq \{x_{k,0}x_{k+1,2}\xi\} \leq c_5 X_{k-1}^{-1}. \end{aligned}$$

The sequence $(X_k)_{k\geq 1}$ grows at least geometrically, so for any pair of integers $j \geq k \geq 1$ such that $j \equiv k \pmod 3$, it follows that

$$|\delta_j - \delta_k| \leq c_5 X_{k-1}^{-1}.$$

Note also that

$$|\{x_{k,0}\xi^3\} - \delta_k| \leq |x_{k,0}\xi^3 - x_{k,2}\xi| \leq c_1|\xi|X_k^{-1}$$

for all $k \geq 1$. Thus $\{x_{k,0}\xi^3\} \to \delta_k$ as $k \to \infty$, so it follows that, for any $i \in \mathbb{Z}/3\mathbb{Z}$, the following limit exists:

$$\theta_i := \lim_{j\to\infty} \{x_{i+3j,0}\xi^3\} = \lim_{j\to\infty} \delta_{i+3j}.$$

Furthermore, whenever $k \equiv i \pmod 3$, then

$$|\theta_i - \{x_{k,0}\xi^3\}| \leq |\theta_i - \delta_k| + |\delta_k - \{x_{k,0}\xi^3\}| \leq (c_5 + c_1|\xi|)X_k^{-1}.$$

Therefore, by Proposition 4.3.2, we have

$$\{x_{k,0}\xi^3\} \geq c_6 X_k^{-1/\gamma^3},$$

for sufficiently large k – specifically, sufficiently large so that $x_{k,0} \neq 0$ for all $k \geq k_0$, for some index k_0. Per Proposition 4.3.4, it follows that for any algebraic integer α of degree at most 3 over $\mathbb{Q}$, we have

$$|\xi - \alpha| \geq c_7 H(\alpha)^{-\gamma^2},$$

for some constant $c_7 > 0$ from which Theorem 1.8 follows. □

Chapter 5

Quartic and higher-degree cases: an open field

Our main focus so far has been algebraic numbers or algebraic integers whose degree is 3 or less. But what about higher-degree cases? That is, suppose that ξ is a real number which is not algebraic number of degree at most $n-1$. Then under what conditions do there exist infinitely many algebraic integers α of degree at most n that approximate ξ "well", namely satisfying the inequality

$$|\xi - \alpha| \leq cH(\alpha)^{-\tau(n)},$$

with $c > 0$ a constant dependent on ξ and n? As we learnt in the three previous chapters, we saw that $\tau(2) = 2$ and $\tau(3) = (3+\sqrt{5})/2$. Furthermore, we also learnt that $\tau(2)$ and $\tau(3)$ are the best possible, by identifying a specific class of extremal real numbers where the aforementioned inequality cannot hold for any α. Little is known about the behaviour of $\tau(n)$ when $n \geq 4$. Therefore, this chapter will focus on giving a brief history and overview of the work done by the mathematical community as of 2015.

To start off, we cite an important theorem proved by Davenport and Schmidt in [DS69], where they applied the ideas from geometry of numbers:

Theorem 5.1 ([DS69, Theorem 2 & Theorem 4]). *Suppose that ξ is a real number which is not algebraic over $\mathbb{Q}$ of degree at most $n-1$. Then there exists infinitely many algebraic integers α of degree at most n so that*

$$|\xi - \alpha| \leq cH(\alpha)^{-\tau(n)},$$

where:

- *c is a constant depending only on n and ξ, and*
- *$\tau(n) : \mathbb{Z}_{\geq 2} \to \mathbb{R}_+$ is a function such that*

$$\tau(n) = \begin{cases} 2 & \text{if } n = 2; \\ \gamma^2 & \text{if } n = 3; \\ 3 & \text{if } n = 4; \\ \left\lfloor \frac{n+1}{2} \right\rfloor & \text{if } n \geq 5. \end{cases}$$

It should be noted that it is unknown whether the values of $\tau(n)$ for $n \geq 4$ are the best possible results. The 1969 Davenport–Schmidt result was improved by Laurent. In the course of proving the following theorem, Laurent also simplified the Davenport–Schmidt argument.

Theorem 5.2 ([Lau03, Theorem])**.** *For any $n \geq 4$, we can take $\tau(n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Roy, in [Roy08], noted that another hurdle is that the only non-trivial upper bound of $\tau(n)$ for $n \geq 4$ that we know of is $\tau(n) \leq n$ (see [Roy08, §1] and the proof of [Bug04, §1, Theorem 3.3] for more information). Roy confessed in [Roy08] that he tried to work on an optimal value of $\tau(4)$ with no success. Yet, this did not stop him from making an educated guess about a possible value for $\tau(4)$. Roy proved that

Theorem 5.3 ([Roy08, Theorem])**.** *Let $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 3$, and let c and λ be positive real numbers. Suppose that, for any sufficiently large value of X, the inequalities*

$$|x_0| \leq X \text{ and } |x_0\xi^j - x_j| \leq cX^{-\lambda} \text{ for } j = 1, 2, 3$$

have a non-zero solution $\mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$. Let

$$\lambda_3 := \frac{1}{2}\left(2 + \sqrt{5} - \sqrt{7 + 2\sqrt{5}}\right) \approx 0.4245.$$

Then $\lambda \leq \lambda_3$. Furthermore, if $\lambda = \lambda_3$, then c is bounded below by a positive constant depending only on ξ.

By using Lemma 1 of [DS69, §2] which we state below, Roy also proved that $\tau(4)$ can be taken to be $\lambda_3^{-1} + 1 \approx 3.3556$. However Roy remarked in the same paper that $\lambda_3^{-1} + 1$ is, unfortunately, not optimal. He did not include the proof of this claim, but only noted that the tools involved in the "quite involved" argument (as characterized by Roy himself), include a variation of the bracket $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ as defined in Definition 3.4.1.

Lemma 5.4. *Assume that $n \geq 2$, $\xi \in \mathbb{R}$ and $\lambda > 0$. Suppose also that for some $c > 0$ there are arbitrarily large values of X such that the inequalities*

$$|x_0| \leq X, \quad |x_0\xi^m - x_m| \leq cX^{-\lambda} \quad (1 \leq m \leq n-1)$$

have no integral solutions $(x_0, x_1, \ldots, x_{n-1})$ not all 0. Then there exist infinitely many algebraic integers α of degree at most n such that

$$0 < |\xi - \alpha| \leq c' H(\alpha)^{-1-1/\lambda},$$

where c' is a constant depending only on n and ξ.

Roy hoped that [Roy08] would lead to a new, generalized class of extremal real numbers for general n. If such a class can be constructed, then there may be some hope for employing a variation of Roy's arguments presented in his two-part papers [Roy04] and [Roy03]; however, progress remains elusive.

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