UNITARY UNTOUCHABLE NUMBERS

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Contents

Abstract

In 1973, Erdős, in [Erd73], proved that a positive proportion of numbers are s-untouchable: that is, not of the form $s(n)$, where $s(n) := \sigma(n) - n$ is the sum of the proper divisors of n. We investigate the analogous question where σ is replaced with similar divisor functions, such as the sum-of-unitary-divisors function σ^* (which sums divisors d of n such that $(d, n/d) = 1$). We use a slightly modified version of Erdős's original argument from the aforementioned paper. In one of the cases, the theory of covering congruences makes a surprising appearance.

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Chapter 1

Introduction

1.1 Arithmetical functions

Any complex-valued function defined on the set of positive integers is called an *arithmetical function*. Some examples include $\sigma(n)$, the-sum-of-divisors function; $\varphi(n)$, Euler's φ-function; and Dirichlet characters. Interests in arithmetical functions by mathematicians can be traced back to as early as 2,500 years ago. Pythagoras already studied $s(n) := \sigma(n) - n$, i.e., the sum-of-proper-divisors function. He was particularly interested in *amicable pairs*: that is, two distinct integers m, n with $s(n) = m$ and $s(m) = n$. On the other hand, Euclid was interested in studying *perfect numbers*, the integers n such that $s(n) = n$.

Since then number theorists studied various properties of arithmetical functions. Some number theorists studied the range of arithmetical functions. For instance, Erdős studied the range of $\varphi(n)$ and proved in [Erd35] that the size of φ's range between 1 and x is $x/(\log x)^{1+o(1)}$. The true magnitude of the range of φ was eventually computed due to efforts of Erdős and Hall (see [EH73] and [EH76]), Maier and Pomerance (see [MP98]), and Ford (see [For98]). However, the asymptotic formula of the range of φ is yet to be computed.

Other number theorists studied what integers *cannot* be in the image of a function. This led to a study of *f-untouchable numbers*, where f is an arithmetical function. We call an integer m *f-untouchable* if there is no n such that $f(n) = m$.

1.2 Definitions

In this section we introduce some important definitions that will be relevant for this dissertation.

Definition 1.1. An arithmetical function $f : \mathbb{Z}_+ \to \mathbb{C}$ is called *multiplicative* if, for any $m, n \in \mathbb{Z}_+$ with $(m, n) = 1$, f satisfies $f(mn) = f(m)f(n)$.

Remark 1.2. The fundamental theorem of arithmetic therefore implies that, if f is multiplicative it suffices to define f for all primes and prime-powers.

Definition 1.3. A divisor d of n is said to be *a unitary divisor of* n if $d \mid n$ and $(d, n/d) = 1$. We write $d \parallel n$ if d is a unitary divisor of n.

Definition 1.4. The function $\sigma(n)$ denotes the sum of divisors of n. That is,

$$\sigma(n) = n \prod_{\substack{p^a \| n \\ p \text{ prime}}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^a}\right) = \prod_{\substack{p^a \| n \\ p \text{ prime}}} (1 + p + p^2 + \cdots + p^a).$$

Definition 1.5. The function $\sigma^*(n)$ denotes the sum of unitary divisors of n. That is,

$$\sigma^*(n) = n \prod_{\substack{p^a \| n \\ p \text{ prime}}} \left(1 + \frac{1}{p^a}\right) = \prod_{\substack{p^a \| n \\ p \text{ prime}}} (1 + p^a)\,.$$

Moreover, we define $s^*(n) := \sigma^*(n) - n$.

Example 1.6. $\sigma(n)$, $\sigma^*(n)$, and $\varphi(n)$ are examples of multiplicative functions.

Definition 1.7. Suppose $A \subseteq \mathbb{Z}_+$. With $a(n) := |\{1, 2, \ldots, n\} \cap A|$, the *lower (asymptotic) density* $\underline{d}(A)$ is defined as

$$\underline{d}(A) = \liminf_{x \to \infty} \frac{a(n)}{n}.$$

Similarly, the *upper (asymptotic) density* $\overline{d}(A)$ is defined as

$$\overline{d}(A) = \limsup_{x \to \infty} \frac{a(n)}{n}.$$

We say that A *has (asymptotic) density* if $\underline{d}(A) = \overline{d}(A)$.

Example 1.8. Let A be the set of integers that are multiples of 5. Then the asymptotic density of A is $1/5$. In general, if A denotes the set of integer multiples of m, then the asymptotic density of A is $1/m$.

Remark 1.9. If A is a finite set, then the asymptotic density of A is 0. However, the converse does *not* hold. Consider P, the set of all prime numbers. Euclid proved that there are infinitely many prime numbers, so $|P| = \infty$. However, the prime number theorem implies that the asymptotic density of P is 0. Therefore, if a set has the asymptotic density 0, then it implies that the elements in that set are sparse in the set of integers; however, it does not say if that set is finite or infinite. While we know by [Wir59] that the set of perfect numbers has density 0, we do not know whether there are infinitely many perfect numbers.

Remark 1.10. Recall that the size of the range of φ between 1 and x is $x/(\log x)^{1+o(1)}$. Therefore, it follows that the range of φ has density 0. Similar argument from [Erd35] can be used to show that the ranges of σ and of σ^* have density 0 as well. Interestingly, however, we see that, in later chapters, that the ranges of s and s^* have a positive lower density.

1.3 Known results about different types of untouchable numbers

One of the key results came first from Erdős, who proved in [Erd73] in 1973, that there are infinitely many s-untouchable numbers (some mathematicians call them *nonaliquot numbers*) by showing that the lower density of s-untouchables is positive. However, he did not provide the explicit lower bound. Herman te Riele computed in [tR76] that the lower density of s-untouchables is greater than 0.0324. Banks and Luca improved this result (see [BL05]) by showing that the lower density of s-untouchables is at least $1/48 + o(1)$ (Given that their result is weaker than te Riele's bound, it is likely that Banks and Luca were unaware of te Riele's thesis by the time they published their paper.). This was not improved until 2011 when Chen and Zhao (see [CZ11]) showed that the lower density of s-untouchables is at least $0.06 + o(1)$.

Herman te Riele also studied s_ψ-untouchable numbers, where ψ is the Dedekind-ψ function defined as

$$\psi(n) := n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right), \text{ and } s_\psi(n) := \psi(n) - n.$$

He proved that a geometric sequence of the form $(2^n \cdot 3 \cdot R)_{n\geq 1}$ for some fixed R such that $(6, R) = 1$ comprises entirely either of s_ψ-untouchables or s_ψ-touchables. Then he proved that $30 = 2 \cdot 3 \cdot 5$ cannot be in the range of $\psi(n) - n$, thereby proving that there are infinitely many s_ψ-untouchable numbers.

Erdős and Sierpiński asked whether there are infinitely many *noncototients*, integers m such that no n can satisfy $n - \varphi(n) = m$. This problem is mentioned in Guy's book in unsolved number theory problems (see **B36** in [Guy04]). Browkin and Schinzel gave an affirmative answer in [BS95] by constructing a family of noncototients, namely the integers of the form $2^k \cdot 509203$. Flammenkamp and Luca (see [FL00]) provided sufficient conditions for $k \in \mathbb{Z}_+$ so that the geometric sequence $(2^m k)_{m \geq 1}$ entirely consists of noncototients. They also provided a few more additional families of noncototients using these conditions.

1.4 Known results on unitary untouchable numbers; statement of the problem

Unitary untouchable numbers – positive integers not in the range of $s^*(n) := \sigma^*(n) - n$ – were first mentioned in Chapter 9 of te Riele's doctoral thesis [tR76]. Guy, from **B10** in [Guy04], also mentioned unsolved problems related to untouchable numbers. Surprisingly, little is known about unitary untouchable numbers. Felice Russo found a handful of untouchable numbers up to 1,000. te Riele computed the unitary untouchables up to 20,000 in [tR76], followed by David Wilson in 2001, who expanded the table up to 100,000 (see [Guy04]). In addition to computational results for $s^*(n)$, te Riele also commented on the relation between de Polignac's conjecture and unitary untouchables. The conjecture is introduced below:

Conjecture (de Polignac, 1849, [dP49]). *Any odd number greater than* 1 *is of the form* $2^k + p$*, where* k *is a positive integer and* p *is either a prime or* 1*.*

te Riele observed that, since $s^*(2^k p) = 2^k + p + 1$, all even numbers greater than 2 are s^*-touchable provided that de Polignac's conjecture is true. In 1950 Erdős [Erd50] and van der Corput [vdC50] proved independently that there is a positive proportion of odd integers that disprove de Polignac's conjecture. te Riele, however, could not prove whether there are infinitely many unitary untouchable numbers. In this thesis, we provide an answer to the question that te Riele left open in his doctoral thesis.

In Chapter 2, we examine Erdős's argument in [Erd73], whose argument we used extensively to tackle the research question. In Chapter 3, we prove that there are infinitely many unitary untouchable numbers. We achieve this by constructing a residue class such that a positive proportion of integers in that residue class are unitary untouchable numbers. The theory of covering congruences and te Riele's astute observation on the relationship between s^*-touchables and de Polignac's conjecture are two key ingredients in constructing this residue class. Finally, we note that we are largely interested in even unitary untouchable numbers. Montgomery and Vaughan, in [MV75], proved that the density of even numbers that Goldbach's conjecture do not hold is 0. Observe that $s^*(pq) = 1+p+q$ for distinct odd primes p and q, and the theorem by Montgomery and Vaughan implies that almost all odd numbers are s^*-touchable.

Chapter 4 deals with the computational aspects of unitary untouchable numbers. In this short chapter, the central relation used in the algorithm is proved, followed by a flowchart depicting the ideas behind the algorithm. Chapter 5 discusses some open problems, including a conjecture based on the computational results. Appendix A contains the Mathematica code used to compute the number of unitary untouchable numbers up to 10^8 and the density; Appendix B contains the data obtained by running the code in Appendix A. Appendix C contains the Mathematica code that computes the number of noncototients up to 10^8; on the other hand, Appendix D contains the Mathematica code that computes the number of s-untouchables, again up to 10^8. Finally, Appendix E provides companion results for noncototients and s-untouchables.

Chapter 2

Erdős's original argument

2.1 Introduction

Erdős proved two central theorems in [Erd73]:

Theorem 2.1. *The lower density of s-untouchable numbers is positive.*

Theorem 2.2. *Let $P_k = 2 \cdot 3 \cdot \cdots \cdot p_k$ be the product of the first k prime numbers. Also, let k be a positive integer and $A(k, x)$ the number of integers n such that*

$$\sigma(n) - n \leq x \text{ and} \tag{2.1}$$

$$\sigma(n) - n \equiv 0 \pmod{P_k}. \tag{2.2}$$

Then for any $\varepsilon > 0$ there exists some k so that $A(k, x)$ is smaller than $\varepsilon x / P_k$ for all $x > x_0(\varepsilon, k)$.

After we establish that Theorem 2.1 follows from Theorem 2.2, we will proceed to proving Theorem 2.2. We ultimately prove that there are infinitely many *even* s-untouchable numbers.[1]

[1] As for odd s-untouchable numbers, it is conjectured that 5 is the only odd s-untouchable number. There

2.2 Relation between Theorem 2.1 and Theorem 2.2

Proposition 2.3. *Theorem 2.1 follows from Theorem 2.2.*

Proof. We shall prove by contradiction. Let ε be an arbitrary real number less than 1; we then choose a sufficiently large k which makes Theorem 2.2 hold. To begin, note that the natural density of integers divisible by P_k is $1/P_k$, and that the upper density of integers m for which (2.1) has a solution (= touchable numbers) is at most ε/P_k. Contrary to what Theorem 2.1 states, suppose that the lower density of s-untouchable number is 0. Then the upper density of touchable numbers is 1. Hence, the upper density of touchable numbers divisible by P_k is exactly $1/P_k$. This is a contradiction, so Theorem 2.1 follows from Theorem 2.2. □

2.3 Proof of Theorem 2.2

To begin with, we note that numbers of the form $\sigma(n)$ are easier to study than numbers of the form $\sigma(n) - n$. In order to compute $A(k, x)$, we will break into three cases:

1. $n \equiv 1 \pmod 2$

2. $n \equiv 0 \pmod 2$ but $n \not\equiv 0 \pmod{P_k}$

3. $n \equiv 0 \pmod{P_k}$

Lemma 2.4 (Case 1)**.** *Define $A_1(k, x)$ as the number of odd integers satisfying* (2.1) *and* (2.2)*. Then for any fixed choice of k we have $A_1(k, x) = o(x)$ as $x \to \infty$.*

Proof. Let $A_1(k, x)$ be the number of integers of the form $\sigma(n) - n$ with n odd and satisfy (2.1) and (2.2). Since k is a positive integer, P_k is even. It follows that $\sigma(n)$ is odd,

is a "proof," but the proof assumes that a slightly modified version of Goldbach's conjecture (any even number larger than 6 can be expressed as the sum of two distinct primes) is true.

which further implies that $n = t^2$ for some odd t. First, we consider when t is a prime. Then $\sigma(n) - n = 1 + t \geq t = \sqrt{n}$, so $\sqrt{n} = t < x$. The number of possible values of n is equal to the number of prime numbers less than or equal to x. Hence, there are $\pi(x)$ choices for t, where $\pi(x)$ is the prime-counting function. Now let t be composite. We note that the smallest prime factor p of t is no greater than $n^{1/4}$; from this it follows that $np^{-1} \geq n^{3/4}$. Thus, $n^{3/4} \leq np^{-1} < \sigma(n) - n \leq x$, or $n < x^{4/3} < x^{3/2}$. Therefore, $t < x^{3/4}$, so $A_1(k, x) < \pi(x) + x^{3/4}$. According to the Prime Number Theorem,

$$\begin{aligned} \pi(x) &\sim \frac{x}{\log x}, \text{ so} \\ \frac{\pi(x)}{x} &\sim \frac{1}{\log x}. \end{aligned}$$

Also,

$$\lim_{x\to\infty} \frac{x^{3/4}}{x} = \lim_{x\to\infty} \frac{1}{x^{1/4}} = 0.$$

From this it is clear that $\pi(x) + x^{3/4} = o(x)$. We thus arrive at $A_1(k, x) = o(x)$. □

Lemma 2.5 (Case 2)**.** *Suppose $A_2(k, x)$ is the number of even integers with $n \not\equiv 0 \pmod{P_k}$ satisfying* (2.1) *and* (2.2)*. Then for any fixed k we have $A_2(k, x) = o(x)$ as $x \to \infty$.*

Proof. First, we let q_i be prime numbers such that $q_i \equiv -1 \pmod{P_k}$, where $i \in \mathbb{Z}_+$. Clearly, $\sum q_i^{-1} = \infty$, according to Dirichlet's theorem on primes in arithmetic progressions. Therefore, if we let $v_i = \frac{q_i - 1}{q_i^2}$, the series $\sum v_i$ is divergent as well. We need the following lemma to proceed further:

Lemma. *Suppose that $a_i \to 0$ as $i \to \infty$ and $0 < a_i < 1$ for all $i \in \mathbb{Z}_+$. Then*

$$\sum_{i=1}^{\infty} a_i = +\infty \textit{ if and only if } \prod_{i=1}^{\infty} (1 - a_i) = 0.$$

Proof. Since a_i is a sequence converging to 0 and $0 < a_i < 1$, for any b with $0 < b < 1$, we can choose, for some large k, $0 < a_i < b$ for $i > k$. Without loss of generality, let $b = 1/2$. We note that

$$-2x < \log(1-x) < -\frac{1}{2}x, \tag{2.3}$$

where $0 < x < 1/2$. This fact is not hard to prove. First, we note that $-2(0) = \log(1-0) = -1/2(0) - 0$. We also note that the value of derivative of $\log(1-x)$ satisfies

$$\frac{d}{dx}(-2x) = -2 < \frac{d}{dx}\log(1-x) = \frac{1}{x-1} < -1 < \frac{d}{dx}\left(-\frac{1}{2}x\right) = -\frac{1}{2}$$

Finally, the value of each function at the other end of the interval is

$$-2 \times \frac{1}{2} = -1 < \log\frac{1}{2} \approx -0.693 < -\frac{1}{2} \times \frac{1}{2} = -0.25.$$

Thus, (2.3) is true, as required.

($\Rightarrow$) We let $\sum a_i = +\infty$. We note that, by (2.3),

$$\log\left(\prod_{i=1}^{k}(1-a_i)\right) = \sum_{i=1}^{k}\log(1-a_i) < -\frac{1}{2}\sum_{i=1}^{k}a_i.$$

It follows that, as $k \to \infty$, $\log(\prod(1-a_i)) \to -\infty$ as $\sum a_i$ approaches infinity. The result thus follows.

($\Leftarrow$) Suppose $\prod(1-a_i) = 0$. Then

$$\log\left(\prod_{i=1}^{k}(1-a_i)\right) \to -\infty \text{ as } k \to \infty.$$

But then we note that

$$-2\sum_{i=1}^{k}a_i < \log\left(\prod_{i=1}^{k}(1-a_i)\right),$$

which implies that $\sum a_i \to +\infty$. $\square$

Since $\prod(1 - v_i) = 0$, for any arbitrary $\eta > 0$, we can choose some large integer r, which satisfies

$$\prod_{i=1}^{r}(1 - v_i) < \eta. \tag{2.4}$$

We note that $\sigma(n) \equiv 0 \pmod{P_k}$ if, for some i, $q_i \mid n$ but $q_i^2 \nmid n$.

Suppose $B_r = \prod_{i=1}^{r} q_i$ and u is a residue class modulo B_r^2 such that, for some $i \le r$, $q_i \mid u$ but $q_i^2 \nmid u$. It follows that if $n \equiv u \pmod{B_r^2}$, then $\sigma(n) \equiv 0 \pmod{P_k}$. Since we are concerned with n such that $\sigma(n) \not\equiv 0 \pmod{P_k}$, we precisely need to compute the number of residue classes modulo B_r^2 which do not satisfy $q_i \parallel u$. Upon noting that v_i signifies the natural density of numbers divisible by q_i but not by q_i^2, we see that the number of desired residue classes is

$$B_r^2 \prod_{i=1}^{r}\left(1 - \frac{q_i - 1}{q_i^2}\right). \tag{2.5}$$

But then according to (2.4), the expression (2.5) is small. In particular,

$$B_r^2 \prod_{i=1}^{r}\left(1 - \frac{q_i - 1}{q_i^2}\right) < \eta(B_r^2).$$

Thus, the density of integers n with $\sigma(n) \not\equiv 0 \pmod{P_k}$ is 0.

Since n is even, the fact that $\sigma(n) - n \le x$ implies $n \le 2x$. Also, we have, by assumption, $n \not\equiv 0 \pmod{P_k}$, which implies $\sigma(n) \not\equiv 0 \pmod{P_k}$. Thus, $A_2(k, x)$ is the number of integers less than or equal to $2x$ with $\sigma(n) \not\equiv 0 \pmod{P_k}$. Thus, we have $A_2(k, x) = o(x)$ as $x \to \infty$, as desired. $\square$

Remark 2.6. In the original paper, Erdős proved a separate lemma, which is the Case 2 Lemma for primes. He proved this in the appendix to preserve the flow of his paper. However, we realize that the proof exactly goes the same way even if p is not a prime. This is so because any number and 1 are co-prime. This means that we can still apply Dirichlet's theorem on primes in arithmetic progressions. The rest of the proof goes exactly the same.

Lemma 2.7 (Case 3)**.** *Let $A_3(k,x)$ be the number of integers divisible by P_k satisfying* (2.1) *and* (2.2)*. Suppose k is greater than k_0, a constant which depends on ε. For any ε and sufficiently large $x > x_0(\varepsilon)$, the number of integers n for which $n \equiv 0 \pmod{P_k}$ and* (2.1) *and* (2.2) *hold is at most $(\varepsilon/2)(x/P_k) + 1$.*

Proof. Since n is divisible by P_k, it follows that

$$\sigma(n) = \sum_{d|n} \frac{n}{d} \geq \sum_{d|P_k} \frac{n}{d} = n \prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right).$$

We also see that for any $\varepsilon > 0$ we can choose $k > k_0(\varepsilon)$ so that

$$n \prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right) > n \left(1 + \sum_{i=1}^{k} \frac{1}{p_i}\right) > n \left(1 + \frac{2}{\varepsilon}\right),$$

since $\sum (1/p_i) = \infty$. Hence,

$$\frac{2}{\varepsilon} n < \sigma(n) - n \leq x, \text{ so } n < \frac{\varepsilon x}{2}.$$

Therefore, it follows that $A_3(k,x)$ is at most $1 + \lfloor \varepsilon x / 2P_k \rfloor \leq 1 + (\varepsilon/2)(x/P_k)$. □

Hence, from the three cases discussed above, it follows that

$$A(k,x) = A_1(k,x) + A_2(k,x) + A_3(k,x) < \frac{\varepsilon x}{2P_k} + 1 + o(x) \leq \frac{\varepsilon x}{P_k},$$

proving Theorem 2.2.

Chapter 3

On unitary untouchable numbers

3.1 Statement of the central theorem

We formally define unitary untouchable numbers and state the central theorem:

Definition 3.1. Suppose $\sigma^*(n)$ is the function as defined in Definition 1.5, and $s^*(n) = \sigma^*(n) - n$. Then $m \in \mathbb{Z}_{\geq 0}$ is called a *unitary untouchable number* if there is no $n \in \mathbb{Z}_+$ such that $s^*(n) = m$.

Theorem 3.2. *The lower asymptotic density of the unitary untouchable numbers is positive. Therefore, there are infinitely many unitary untouchable numbers.*

Our strategy will be as follows.

- Break into the following cases:

  0. $n = 2^w p$ ($w \geq 1, p$ odd prime)

  I. $n \equiv 2 \pmod 4$

  II. $n = 2^w p^a$ ($w > 1, a > 1, p$ odd prime)

  III. $n \equiv 0 \pmod 4$, with more than one distinct odd prime factor

IV. $n \equiv 1 \pmod 2$

- Construct a residue class that contains no number of the form $s^*(n)$ for n in Case 0,
- Show that the residue class constructed in Case 0 has a positive proportion of integers not of the form $s^*(n)$ for all $n \equiv 2 \pmod 4$ (Case I)
- Prove that the set of $s^*(n)$ for n in Cases II, III, and IV which fall into the residue class constructed in Case 0 has density 0.

These steps allow us to conclude that a positive proportion of the residue class constructed in Case 0 is unitary untouchable, which also implies that the lower density of unitary untouchable numbers is positive.

Remark 3.3. We would like to comment why a different approach is necessary in the case of $2^w p$. Unlike Case II, there are "too many" integers of the form $2^w p$ with $s^*(2^w p) = 1+2^w+p \le x$. To examine it at a more rigorous level, consider the following two inequalities:

$$2^w \le \frac{x}{\log x} \text{ and} \tag{3.1}$$

$$p \le x - 1 - \frac{x}{\log x}. \tag{3.2}$$

From (3.1) we have $w \le \log(x/\log x)/\log 2$. By (3.2) there are $\pi(x - 1 - x/\log x)$ choices of p, where $\pi(x)$ denotes the prime-counting function. It follows from the prime number theorem that

$$\frac{\log x - \log\log x}{\log 2} \cdot \pi\left(x - 1 - \frac{x}{\log x}\right) \sim \frac{\log x - \log\log x}{\log 2} \cdot \frac{x - 1 - x/\log x}{\log(x - 1 - x/\log x)}.$$

Finally, observe that

$$\lim_{x\to\infty} \frac{\frac{\log x - \log\log x}{\log 2} \cdot \frac{x-1-x/\log x}{\log(x-1-x/\log x)}}{x} = \frac{1}{\log 2} \lim_{x\to\infty} \frac{(\log x - \log\log x)(x-1-x/\log x)}{x\log(x-1-x/\log x)}$$
$$= \frac{1}{\log 2} > 0,$$

by L'Hôpital's rule.

3.2 Proof of Theorem 3.5

3.2.1 "Case 0" and the construction of Theorem 3.5

To begin with, we shall prove a lemma that will help us construct a residue class that does not contain any integer of the form $s^*(2^w p)$ for any $w \geq 1$ and odd prime p. We use the theory of covering congruences to achieve this.

Lemma 3.4 ("Case 0")**.** *Let $n = 2^w p$, with $w \geq 1$ and p an odd prime. Then there exist c and odd d such that $s^*(n) \not\equiv c \pmod{d}$ for any w and p.*

Proof. It is easy to verify that any $w \in \mathbb{Z}$ satisfies at least one of the following congruences:

$$w \equiv 1 \pmod{2}, \quad w \equiv 1 \pmod{3}, \tag{3.3}$$
$$w \equiv 2 \pmod{4}, \quad w \equiv 4 \pmod{8}, \tag{3.4}$$
$$w \equiv 8 \pmod{12}, \quad w \equiv 0 \pmod{24}. \tag{3.5}$$

For each modulus $m \in \{2, 3, 4, 8, 12, 24\}$ we find an odd prime q such that $2^m \equiv 1 \pmod{q}$. With this we compute what conditions $N := 1 + 2^w + p$ *cannot* satisfy if w is in one of the above six congruence classes. We shall demonstrate this with one of the moduli. Suppose $m = 8$. Since $2^4 \equiv -1 \pmod{17}$, it follows that the order of 2 is indeed 8 in $(\mathbb{Z}/17\mathbb{Z})^*$. Hence, let $q = 17$. In the case of $m = 8$, we are only dealing with integers $w \equiv 4 \pmod 8$, so we have $2^w \equiv 2^4 \equiv -1 \pmod{17}$. Thus we have $N \equiv 0 + p \pmod{17}$; and since $N > 1 + 2^4 = 17$, we know that $N \not\equiv 0 \pmod{17}$. We will apply this idea to other moduli. Then we have

m	q	$2^w \bmod q$	$N \bmod q$	Conclusion:
2	3	2	$N \equiv p$	$N \not\equiv 0 \pmod 3$ or $p = 3$
3	7	2	$N \equiv 3 + p$	$N \not\equiv 3 \pmod 7$ or $p = 7$
4	5	-1	$N \equiv p$	$N \not\equiv 0 \pmod 5$ or $p = 5$
8	17	-1	$N \equiv p$	$N \not\equiv 0 \pmod{17}$ or $p = 17$
12	13	-4	$N \equiv -3 + p$	$N \not\equiv -3 \pmod{13}$ or $p = 13$
24	241	1	$N \equiv 2 + p$	$N \not\equiv 2 \pmod{241}$ or $p = 241$

Upon applying the Chinese Remainder Theorem to the six residue classes in the last column, i.e.,

$$N \equiv 0 \pmod 3, \quad N \equiv 0 \pmod 5, \tag{3.6}$$

$$N \equiv 3 \pmod 7, \quad N \equiv -3 \pmod{13}, \tag{3.7}$$

$$N \equiv 0 \pmod{17}, \quad N \equiv 2 \pmod{241}, \tag{3.8}$$

we obtain the residue class $-1518780 \pmod{3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241}$. Let $c = -1518780$ and $d = 5592405$. If p does not match with any of the six q's, then $s^*(2^w p)$ cannot satisfy any of the congruences in (3.6) as w is covered by the congruences in (3.3), from which it follows that $s^*(2^w p) \not\equiv c \pmod d$. Thus, we are only left with cases when p matches with one of the six q's.

If w is odd, then $p = 3$. We need $N = 2^w + 4 \equiv 3 \pmod 7$, but the congruence $2^w \equiv -1 \pmod 7$ has no solution.

Suppose $w \equiv 1 \pmod 3$; therefore $p = 7$ and $N = 2^w + 8$. Additionally, we assume that w is even, from the prior case. Then $2^w \equiv 1$ or $4 \pmod 5$, from which it follows that

$N \equiv 4$ or $2 \pmod 5$, which contradicts $N \equiv 0 \pmod 5$.

For $w \equiv 2 \pmod 4$ so that $p = 5$ and $N = 2^w + 6$, noting $2^w \equiv 1 \pmod 3$ leads us to conclude that $N \equiv 1 \pmod 3$, contradicting $N \equiv 0 \pmod 3$.

If $w \equiv 4 \pmod 8$, then $p = 17$ and $N = 2^w + 18$. But then $2^w \not\equiv 0 \pmod 3$, so $N \not\equiv 0 \pmod 3$ either.

Let $w \equiv 8 \pmod{12}$. Then $p = 13$ and $N = 2^w + 14$. Upon noting $2^{12} \equiv 1 \pmod 7$ and $2^8 \equiv 4 \pmod 7$, we conclude that $N \equiv 4 \pmod 7$, which contradicts $N \equiv 3 \pmod 7$.

Finally, suppose $w \equiv 0 \pmod{24}$. In this case, we have $p = 241$ and $N = 2^w + 242$. Since $2^w \equiv 1 \pmod 7$, we get $N \equiv 243 \pmod 7$; but then $243 \not\equiv 3 \pmod 7$, a contradiction.

This concludes the proof of the proposition. □

Now that we defined c and d, we are ready to state the theorem that we intend to show in order to prove Theorem 3.2:

Theorem 3.5. *Denote $P_k = 2 \cdot 3 \cdot \dots \cdot p_k$, i.e., the product of the first k primes, and define $Q_k := P_k/(d, P_k)$. Let r_k be the residue class modulo dQ_k^2 that satisfies*

$$\sigma^*(n) - n \equiv Q_k \pmod{Q_k^2} \text{ and}$$
$$\sigma^*(n) - n \equiv c \pmod d.$$

There exist k_0 and x_0 such that if $k > k_0$ and $x > x_0(k)$ then the number of integers n satisfying

$$\sigma^*(n) - n \leq x \text{ and} \tag{3.9}$$
$$\sigma^*(n) - n \equiv r_k \pmod{dQ_k^2} \tag{3.10}$$

is less than $x/2dQ_k^2$.

Corollary 3.6. *The lower density of unitary untouchable numbers in the residue class* r_k *mod* dQ_k^2 *is at least* $1/2dQ_k^2$ *for a sufficiently large* k.

Corollary 3.7. *Theorem 3.2 follows from Theorem 3.5.*

3.2.2 Case I

Before proving the main result for Case I, we need the following lemma, which is the special case of Lemma 9.2 from [tR76]:

Lemma 3.8 (te Riele)**.** *For any* $z \in \mathbb{Z}_+$ *the number of positive integers* $n \leq x$ *such that* $z \nmid \sigma^*(n)$ *is* $o(x)$ *for* $x \to \infty$.

Proof. One can apply the ideas used to prove Lemma 2.5. Alternately, refer to [Sco73], which proved the general case of te Riele's Lemma 9.2. □

Lemma 3.9 (Case I)**.** *Suppose* $n \equiv 2 \pmod 4$, *i.e.,* 2 *is a unitary divisor of* n.
Then for any $\varepsilon > 0$ *there exist* $k_0(\varepsilon)$ *and* $x_0(\varepsilon)$ *such that if* $k > k_0(\varepsilon)$, *then the number of* n *satisfying* (3.9) *and* (3.10) *is at most* $1 + \varepsilon x/dQ_k^2$ *for all* $x > x_0(\varepsilon)$.

Proof. We note that we have

$$x \geq \sigma^*(n) - n = 3 \prod_{\substack{p^l \| n \\ p \text{ odd prime}}} \left(1 + p^l\right) - n \geq \frac{3}{2}n - n = \frac{1}{2}n.$$

Thus we have $\prod_{\substack{p^l \| n \\ p \text{ odd prime}}} p^l = \frac{n}{2} \leq x$, or $n \leq 2x$. We can now apply Lemma 3.8 to derive that the number of $n \leq 2x$ such that $dQ_k^2 \nmid \sigma^*(n)$ is $o(x)$ as $x \to \infty$. Hence, it suffices to examine integers n with $dQ_k^2 \mid \sigma^*(n)$. For such n, (3.10) implies $n \equiv -Q_k \pmod{Q_k^2}$, so that $Q_k \parallel n$. Pick an arbitrary $\varepsilon > 0$.

Then we can pick a sufficiently large k so that we have the following:

$$\begin{aligned}\sigma^*(n) &= n \prod_{\substack{q \mid Q_k \\ q \text{ prime}}} \left(1+\frac{1}{q}\right) \prod_{\substack{p^l \| \frac{n}{Q_k} \\ p \text{ prime}}} \left(1+\frac{1}{p^l}\right) \\ &\geq n \prod_{\substack{q \mid Q_k \\ q \text{ prime}}} \left(1+\frac{1}{q}\right) > n \left(1 + \sum_{\substack{q \mid Q_k \\ q \text{ prime}}} \frac{1}{q}\right) > n\left(1+\frac{1}{\varepsilon}\right),\end{aligned}$$

since $\sum(1/q) \to \infty$ as $k \to \infty$. Thus, We have, by (3.9), that

$$\frac{n}{\varepsilon} < \sigma^*(n) - n \leq x, \text{ or } n < \varepsilon x.$$

Finally, we are only concerned with integers $n \equiv -r_k \pmod{dQ_k^2}$. There are at most $\lfloor \varepsilon x/dQ_k^2 \rfloor$ intervals of the length dQ_k^2 up to εx, and for each interval there can only be one integer in the residue class $-r_k \bmod dQ_k^2$. For the leftover partial interval, there can be at most one integer. Thus, there are at most $1 + \lfloor \varepsilon x/dQ_k^2 \rfloor \leq 1 + \varepsilon x/dQ_k^2$ integers. □

3.2.3 Cases II, III, and IV

Lemma 3.10 (Case II)**.** *Let n be an integer of the form $2^w p^a$, with $w > 1, a > 1$ with p an odd prime. Then the density of such integers with $s^*(n) \leq x$ is $o(x)$ for $x \to \infty$.*

Proof. We shall simply count the number of n's. To begin with, we have

$$p^a \leq s^*(n) = 1 + 2^w + p^a \leq x \text{ and} \tag{3.11}$$

$$2^w \leq s^*(n) = 1 + 2^w + p^a \leq x. \tag{3.12}$$

Let $P(x)$ be the number of available p's. As well, we shall define $A(x)$ and $W(x)$ for a and w, respectively. From (3.11) we get $p \leq x^{1/a} \leq x^{1/2}$ and $a \leq \log_p x \leq \log_3 x$; hence,

$A(x) = O(\log x)$ and $p \le x^{1/2}$. It follows that $P(x) = \pi(x^{1/2})$. Finally, by (3.12) we get $w \le \log_2 x$, so $W(x) = O(\log x)$. We observe that

$$\lim_{x\to\infty} \frac{P(x)\cdot A(x)\cdot W(x)}{x} = \lim_{x\to\infty} \frac{\pi(x^{1/2})\cdot O((\log x)^2)}{x} = \lim_{x\to\infty} \frac{\pi(x^{1/2})}{x^{1/2}} \cdot \frac{O((\log x)^2)}{x^{1/2}} = 0.$$

This completes the proof. □

Lemma 3.11 (Case III)**.** *If* $n \equiv 0 \pmod 4$ *and satisfies the condition* (3.10)*, then* n *cannot have more than one distinct odd prime factor.*

Proof. Write $n = 2^e p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, with $e \ge 2$. By assumption, n must satisfy $\sigma^*(n) - n \equiv 2 \pmod 4$, i.e., $\sigma^*(n) = (1+2^e)(1+p_1^{a_1})\dots(1+p_r^{a_r}) \equiv 2 \pmod 4$. We note that $1 + p_i^{a_i}$ are even for all $1 \le i \le r$; thus, if $r \ge 2$, then $\sigma^*(n)$ is divisible by 4. Therefore, $r = 1$, as desired. □

Lemma 3.12 (Case IV)**.** *There are no odd integers* n *satisfying* (3.9) *and* (3.10)*.*

Proof. If n is odd and $n > 1$, then there exist $a \ge 1$ and odd p such that $p^a \parallel n$. Thus, we have $(1 + p^a) \mid \sigma^*(n)$. Hence, $s^*(n)$ is always odd unless $n = 1$. But then $\sigma^*(1) - 1 = 0 \not\equiv 2 \pmod 4$, so we are done. □

3.2.4 Final step

The particular residue class we constructed cannot be of the form $s^*(n)$ if n is an integer in Case 0. Also, Lemmas 3.10, 3.11, and 3.12 imply that the asymptotic density of integers of the form $s^*(n)$ satisfying (3.10) is 0. We are thus left with integers $n \equiv 2 \pmod 4$. We do not want the small numbers of integers $s^*(n)$ in Case II to cause the number of integers $s^*(n)$ that satisfy (3.9) and (3.10) to exceed $x/2dQ_k^2$, so we shall choose ε (the same ε from Lemma 3.9) to be less than 1/2.

Let $\varepsilon = 1/3$. Then the same lemma implies that one can choose sufficiently large k so that the asymptotic density of integers $s^*(n)$ with (3.10) is at most $1/3dQ_k^2$. Hence, the density of unitary untouchable numbers is at least $2/3dQ_k^2$. This completes the proof of Theorem 3.5, and, consequently, of Theorem 3.2.

Chapter 4

The enumeration of unitary untouchable numbers

In this chapter we focus on calculating the density of the set of unitary untouchable numbers. We introduce the algorithm used and ideas behind the algorithm.

Proposition 4.1. *Let m be odd and $j \in \mathbb{Z}_+$. Then*

(i) $s^*(2^j m) = 2^j s^*(m) + \sigma^*(m)$

(ii) $s^*(2^{j+1} m) = 2s^*(2^j m) - \sigma^*(m)$.

Proof. (i) $s^*(2^j m) = \sigma^*(2^j m) - 2^j m = (1 + 2^j)\sigma^*(m) - 2^j m = \sigma^*(m) + 2^j(\sigma^*(m) - m) = 2^j s^*(m) + \sigma^*(m)$.
(ii) $2s^*(2^j m) - \sigma^*(m) = 2(\sigma^*(2^j m) - 2^j m) - \sigma^*(m) = 2[(1 + 2^j)\sigma^*(m) - 2^j m] - \sigma^*(m) = (2 + 2^{j+1})\sigma^*(m) - 2^{j+1} m - \sigma^*(m) = 2^{j+1} s^*(m) + \sigma^*(m) \stackrel{(i)}{=} s^*(2^{j+1} m)$. □

Using the above relation, we construct the following algorithm. The program was written in Mathematica; the code is given in Appendix A. The flowchart below describes how the code works. See Appendix B for the number of unitary untouchable numbers up to 10^8.

3 needs to be added at the end as 3, 5, and 7 are conjectured to be the only odd unitary untouchable numbers. We also note that 3, 5, and 7 are the only odd unitary untouchable numbers up to 10^8, as Goldbach's conjecture has been verified up to a very large number.

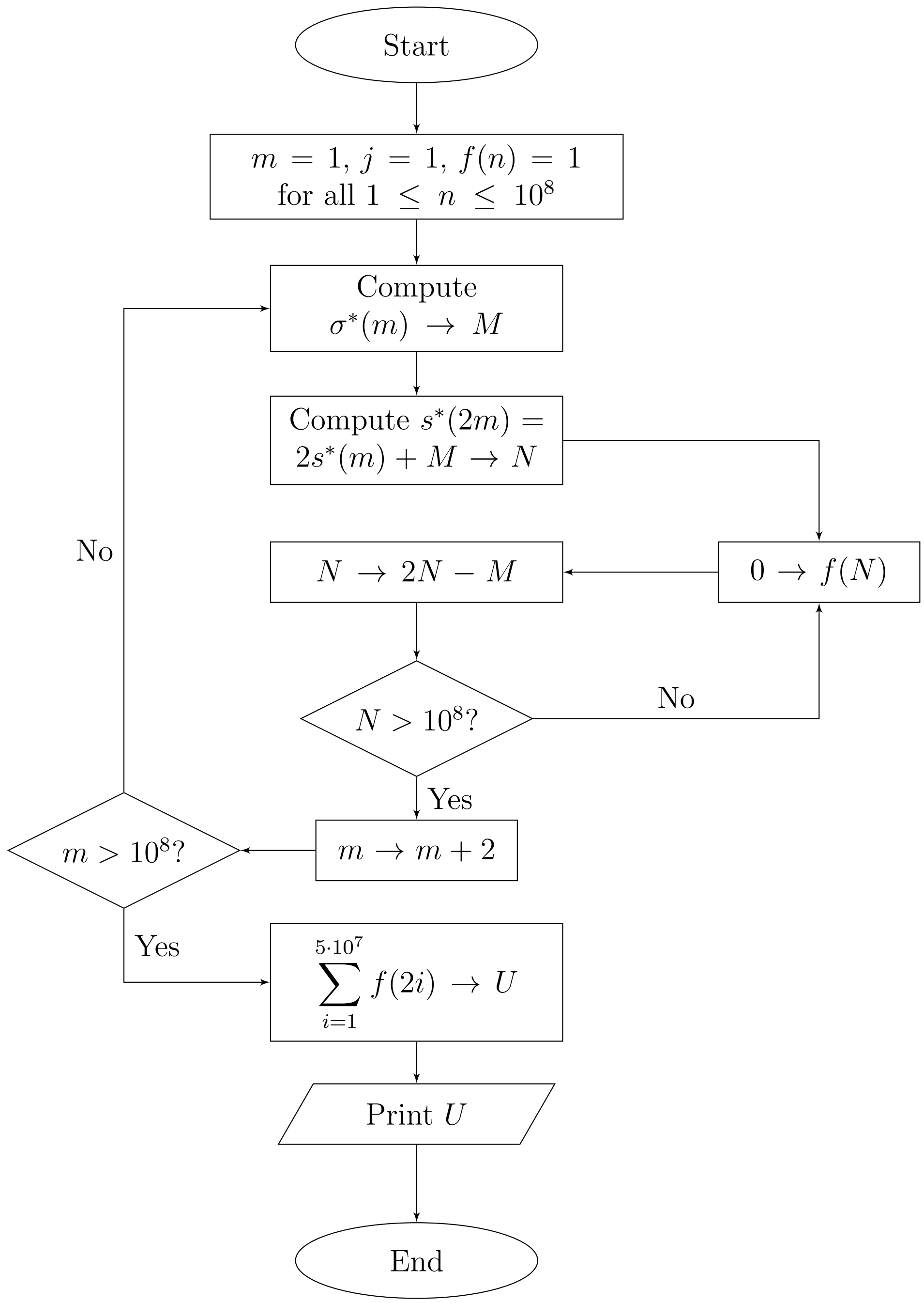

Figure 4.1: Flowchart of the algorithm computing the number of even unitary untouchable numbers $\leq 10^8$

Chapter 5

Open problems

While we proved that there are infinitely many unitary untouchable numbers, there are still open problems related to unitary untouchable numbers. We also introduce other related problems.

I. We have not calculated an explicit lower bound of the lower density of unitary untouchable numbers. In fact, it is not even known if the set of unitary untouchable numbers has asymptotic density. Computational data (see Appendix B) seem to indicate that the following conjecture may be true:

   Conjecture. *The asymptotic density of the set of unitary untouchable numbers exists, and the density is about* 0.01*.*

II. Can one construct a more efficient algorithm that computes the number of unitary untouchable numbers up to some large number? Expand the table given in Appendix B.

III. Can one prove that the lower density of the following sets is positive: even cototients, even s-touchables, or unitary touchables?

Appendix A

Mathematica code for unitary untouchable numbers

```
UnitaryDivisorSigma[n_Integer] := Times @@ (1 + (Power @@@ FactorInteger[n]));
        (* Define $\sigma^*(n)$. *)
UnitaryProperSigma[n_Integer] := UnitaryDivisorSigma[n] - n;
        (* Define $s^*(n)$. *)
Y = 100000000;        (* Want to know how many uu’s there are up to 100000000. *)
v = { };                (* List defined to build a table (See Appendix B). *)
For[Z = 1000000, Z <= Y, Z = Z + 1000000,
        (* Start at 1 000 000, with an increment of 1 000 000. *)
 For[m = 1, m < Z, m++, f[m] = 1];
         (* Let $f(m)$ [the indicator function] be all 1 for now.*)
 For[m = 1, m < Z, m = m + 2,  (* Need to run this for all odd less than Z. *)
  If[m == 1, M = 1, M = UnitaryDivisorSigma[m]];
          (* We have UnitaryDivisorSigma[1] != 1. Force it to be 1. *)
  If[m == 1, P = 0, P = UnitaryProperSigma[m]];
          (* We have UnitaryProperSigma[1] != 0. Force it to be 0. *)
```

```
 Q = 2*P + M;        (* Application of Prop 4.1 (i) *)
 f[Q] = 0;           (* $Q$ is touchable: make $f(Q) = 0$. *)
 While[1 < Q < Z,        (* do this while $1 < Q  < Z$ *)
  Q = 2*Q - M;           (* because this is strictly increasing. *)
  f[Q] = 0               (* Make $f(Q) = 0$ as $Q$ is $s^*$-touchable. *)
  ];
 ];
v = Append[v, {Z, 3 + Sum[f[2*j], {j, 1, Z/2}],
   ScientificForm[(3.0 + Sum[f[2*j], {j, 1, Z/2}])/Z]}];
   ]        (* Upper bound, the number of $s^*$-untouchables, density *)
Export["AppendixB.tex", TableForm[v], "TeX"];
PrintForm[v];
```

Appendix B

The number of unitary untouchable numbers up to 100,000,000

$N(x)$ denotes the number of unitary untouchable numbers up to x. D denotes the density of the set of unitary untouchable numbers.

x	$N(x)$	Δ	$100D(x)$	x	$N(x)$	Δ	$100D(x)$
1000000	9903	9903	0.9903	11000000	111203	10173	1.01094
2000000	19655	9752	0.98275	12000000	121524	10321	1.0127
3000000	29700	10045	0.99	13000000	131827	10303	1.01405
4000000	40302	10602	1.00755	14000000	142447	10620	1.01748
5000000	50081	9779	1.00162	15000000	152930	10483	1.01953
6000000	60257	10176	1.00428	16000000	163766	10836	1.02354
7000000	70518	10261	1.0074	17000000	174187	10421	1.02463
8000000	80987	10469	1.01234	18000000	183664	9477	1.02036
9000000	91087	10100	1.01208	19000000	193361	9697	1.01769
10000000	101030	9943	1.0103	20000000	203113	9752	1.01557

x	$N(x)$	Δ	$100D(x)$	x	$N(x)$	Δ	$100D(x)$
21000000	212971	9858	1.01415	46000000	468094	10322	1.0176
22000000	222825	9854	1.01284	47000000	478644	10550	1.01839
23000000	232812	9987	1.01223	48000000	489027	10383	1.01881
24000000	242928	10116	1.0122	49000000	499571	10544	1.01953
25000000	252997	10069	1.01199	50000000	509695	10124	1.01939
26000000	263322	10325	1.01278	51000000	520265	10570	1.02013
27000000	273594	10272	1.01331	52000000	530726	10461	1.02063
28000000	283985	10391	1.01423	53000000	541225	10499	1.02118
29000000	294329	10344	1.01493	54000000	551825	10600	1.0219
30000000	304631	10302	1.01544	55000000	562362	10537	1.02248
31000000	314927	10296	1.01589	56000000	573087	10725	1.02337
32000000	325368	10441	1.01678	57000000	583702	10615	1.02404
33000000	335888	10520	1.01784	58000000	594241	10539	1.02455
34000000	346027	10139	1.01773	59000000	604709	10468	1.02493
35000000	356024	9997	1.01721	60000000	615349	10640	1.02558
36000000	365843	9819	1.01623	61000000	626200	10851	1.02656
37000000	375833	9990	1.01576	62000000	637111	10911	1.0276
38000000	385954	10121	1.01567	63000000	647753	10642	1.02818
39000000	395990	10036	1.01536	64000000	658513	10760	1.02893
40000000	405978	9988	1.01495	65000000	669526	11013	1.03004
41000000	416147	10169	1.01499	66000000	680386	10860	1.03089
42000000	426428	10281	1.0153	67000000	691106	10720	1.0315
43000000	436923	10495	1.0161	68000000	700849	9743	1.03066
44000000	447512	10589	1.01707	69000000	710781	9932	1.03012
45000000	457772	10260	1.01727	70000000	720741	9960	1.02963

x	$N(x)$	Δ	$100D(x)$	x	$N(x)$	Δ	$100D(x)$
71000000	730628	9887	1.02905	86000000	882395	10318	1.02604
72000000	740672	10044	1.02871	87000000	892885	10490	1.0263
73000000	750700	10028	1.02836	88000000	903123	10238	1.02628
74000000	760626	9926	1.02787	89000000	913463	10340	1.02636
75000000	770576	9950	1.02743	90000000	923994	10531	1.02666
76000000	780418	9842	1.02687	91000000	934445	10451	1.02686
77000000	790537	10119	1.02667	92000000	944737	10292	1.02689
78000000	800642	10105	1.02646	93000000	955175	10438	1.02707
79000000	810790	10148	1.02632	94000000	965478	10303	1.0271
80000000	821201	10411	1.0265	95000000	975773	10295	1.02713
81000000	831481	10280	1.02652	96000000	986255	10482	1.02735
82000000	841599	10118	1.02634	97000000	996795	10540	1.02762
83000000	851758	10159	1.02621	98000000	1007276	10481	1.02783
84000000	861893	10135	1.02606	99000000	1017844	10568	1.02813
85000000	872077	10184	1.02597	100000000	1028263	10419	1.02826

Appendix C

Mathematica code for noncototients

C.1 Proof of the relation used in the algorithm

Proposition C.1. *Let $s_\varphi(n) := n - \varphi(n)$. Suppose also that m is odd and $j \in \mathbb{Z}_+$. Then the following statements hold:*

(i) $s_\varphi(2m) = 2m - \varphi(m)$

(ii) $s_\varphi(2^{j+1}m) = 2s_\varphi(2^j m)$.

Proof. (i) $s_\varphi(2m) = 2m - \varphi(2m) = 2m - \varphi(2)\varphi(m) = 2m - \varphi(m)$.
(ii) $2s_\varphi(2^j m) = 2(2^j m - \varphi(2^j m)) = 2(2^j m - \varphi(2^j)\varphi(m)) = 2(2^j m - 2^{j-1}\varphi(m)) = 2^{j+1}m - 2^j\varphi(m) = 2^{j+1} - \varphi(2^{j+1})\varphi(m) = 2^{j+1}m - \varphi(2^{j+1}m) = s_\varphi(2^{j+1}m)$. □

C.2 Mathematica code

```
Y = 100000000;  (* Want the table up to 100000000. *)
v = {};         (* List for creating a table. *)
For[Z = 10000000, Z <= Y, Z = Z + 10000000,
 For[m = 1, m <= Z, m++, f[m] = 1];
```

```
        (* Let $f(m)$ [the indicator function] be all 1 for now. *)
 For[m = 1, m < Z, m = m + 2,  (*For all odd numbers from 1 to $Z$ *)
  M := EulerPhi[m];             (* Compute $\varphi(m)$. *)
  Q := 2*m - M;                 (* Proposition C.1 (i). *)
  f[Q] = 0;                     (* $Q$ is a cototient; mark it as such. *)
  While[1 <= Q < Z,             (* Do it for all $1 <= Q < Z$ *)
   Q = 2*Q;                     (* Proposition C.1 (ii). *)
   f[Q] = 0                     (* $Q$ is a cototient; mark it as such. *)
   ];
  ];
 v = Append[
   v, {Z, Sum[f[2*j], {j, 1, Z/2}], (Sum[f[2*j], {j, 1, Z/2}] + 0.0)/
     Z}];
            (* Upper bound, the number of noncototients, density. *)
 Export["AppendixC.tex", TableForm[v], "TeX"];
 ]
Print[TableForm[v]]
```

Appendix D

Mathematica code for s-untouchables

D.1 Proof of the relation used in the algorithm

Proposition D.1. *Let $s(n) := \sigma(n) - n$. Suppose also that m is odd and $j \in \mathbb{Z}_+$. Then the following statements hold:*

(i) $s(2m) = 3\sigma(m) - 2m$

(ii) $s(2^{j+1}m) = 2s(2^j m) + \sigma(m)$.

Proof. (i) $s(2m) = \sigma(2m) - 2m = \sigma(2)\sigma(m) - 2m = 3\sigma(m) - 2m$.
(ii) $2s(2^j m) + \sigma(m) = 2(\sigma(2^j m) - 2^j m) + \sigma(m) = 2\sigma(2^j m) + \sigma(m) - 2^{j+1}m = 2\sigma(2^j)\sigma(m) - 2^{j+1}m = 2((1+2+2^2+\cdots+2^j)\sigma(m)) + \sigma(m) - 2^{j+1}m = (1+2+\cdots+2^{j+1})\sigma(m) - 2^{j+1}m = \sigma(2^{j+1})\sigma(m) - 2^{j+1}m = \sigma(2^{j+1}m) - 2^{j+1}m = s(2^{j+1}m)$. □

D.2 Mathematica code

```
Y = 100000000;         (* We want the table up to $10^8$. *)
v = {};        (* List used for creating the table *)
For[Z = 10000000, Z <= Y, Z = Z + 10000000,
```

```
       (* Go to up 100000000, with increment by 10000000 *)
 For[m = 1, m < Z, m++, f[m] = 1];
        (* Let $f(m)$ [the indicator function] be all 1 for now. *)

 (* Recall from the Erdos paper that if $s(n) <= x$ with $n$ odd and $s(n)$ even,
 Then $n = j^2$ where $j$ is odd. If $j$ is composite then we have $n < x^{4/3}$,
 or $j < x^{2/3}$. Note that the prime numbers in this range will be dealt in the
 main loop. We verify all the odd squares $j^2$ with $j$ composite, where
 $\lfloor\sqrt{Z+1}\rfloor < j < Z^{2/3}$, i.e., the range that is not considered
 in the main loop. *)
 R = Floor[Sqrt[Z + 1]]; (* Integer part of $\sqrt{Z+1}$. *)
 If[Mod[R, 2] == 0, R = R + 1];
      (* If the integer part is even, add 1. This makes the starting point odd. *)
 For[j = R, j < Z^(2/3), j = j + 2,
      (* All odd numbers $j^2$ such that $Z+1 < j^2 < Z^{4/3}$)
  If[PrimeQ[j] != True,
      (* The prime cases will be dealt in the main loop, so leave them out. *)
   Q = DivisorSigma[1, j^2] - j^2; (* Calculate $s(j^2)$. *)
   f[Q] = 0; (* $Q$ is $s$-touchable, so make $f(Q) = 0$. *)
   ]
  ];
(* This marks the beginning of the main loop. Now we consider all the odd numbers
between 1 and $Z$. *)
 For[m = 1, m < Z, m = m + 2,
  Q = DivisorSigma[1, m] - m; (* Compute $s(m)$. *)
  P = DivisorSigma[1, m];     (* Compute $\sigma(m)$. *)
  f[Q] = 0;                   (* $Q$ is $s$-touchable; make $f(Q) = 0$. *)
  If[PrimeQ[m], f[m + 1] = 0];
```

```
        (* If $m$ happens to be a prime, then $m+1$ is touchable.
        This is so because $s(m^2) = 1 + m$. Rather than waiting
        for $m^2$ to appear, mark it touchable now. This is necessary
        as $m^2$ can exceed $Z$ while $s(m^2)$ does not.*)

  (* If $s(m) > m+1$ and $s(m)$ is even, then $m$ is a composite square.
  If so, then mark $Q$ touchable, and move on to the next odd number.
  Note that $s(2^j m)$ will be odd for all $j > 0$. *)
  If [Q > m + 1 && Mod[Q, 2] == 0,
   f[Q] = 0,
   (* Otherwise, we iterate $s(2^j m)$ for all $j = 1, 2, ...$ until $s(2^j m) > Z$.
   Note that $s(2^j m)$ is a strictly increasing function as $j$ increases. *)
   Q = 3*P - 2*m;    (* Proposition D.1 (i) *)
   f[Q] = 0;         (* $Q$ is $s$-touchable; mark it as such. *)
   While[1 < Q < Z,  (* Do it while $1 < Q < Z$ *)
    Q = 2*Q + P;     (* Proposition D.1 (ii) *)
    f[Q] = 0;        (* Mark $Q$ as an $s$-touchable *)
    ];  ];  ];
 v = Append[
   v, {Z, 1 + Sum[f[2*j], {j, 1, Z/2}],
    ScientificForm[(1.0 + Sum[f[2*j], {j, 1, Z/2}])/Z]}];
       (* Upper bound, the number of $s$-untouchables, density.
       Add 1 as 5 is the conjectured to be the only odd $s$-untouchable number.
       No need to worry about any other odd numbers as Goldbach's conjecture
       has been verified up to a very large number. *)
 Export["AppendixD.tex", TableForm[v], "TeX"];
 ]
Print[TableForm[v]]
```

Appendix E

The number of noncototients and s-untouchables up to 100,000,000

$N_\varphi(x)$ denotes the number of noncototients up to x. As with Appendix B, D denotes the density of the set of noncototients.

x	$N_\varphi(x)$	Δ	$D(x)$	x	$N_\varphi(x)$	Δ	$D(x)$
100000	10527	10527	0.10527	3000000	335920	112583	0.111973
200000	21433	10906	0.107165	4000000	448955	113035	0.112239
300000	32497	11064	0.108323	5000000	561850	112895	0.11237
400000	43559	11062	0.108898	6000000	674884	113034	0.112481
500000	54757	11198	0.109514	7000000	788080	113196	0.112583
600000	65938	11181	0.109897	8000000	901478	113398	0.112685
700000	77115	11177	0.110164	9000000	1014711	113233	0.11274
800000	88306	11191	0.110383	10000000	1128160	113449	0.112816
900000	99554	11248	0.110616	20000000	2262697	1134537	0.113135
1000000	110786	11232	0.110786	30000000	3398673	1135976	0.113289
2000000	223337	112551	0.111669	40000000	4534957	1136284	0.113374

x	$N_\varphi(x)$	Δ	$D(x)$	x	$N_\varphi(x)$	Δ	$D(x)$
50000000	5671818	1136861	0.113436	80000000	9081939	1137103	0.113524
60000000	6808454	1136636	0.113474	90000000	10218937	1136998	0.113544
70000000	7944836	1136382	0.113498	100000000	11355049	1136112	0.11355

The table below is the companion result for s-untouchables. $N_\sigma(x)$ denotes the number of s-untouchables up to x. Similarly, D denotes the density of the set of s-untouchables.

x	$N_\sigma(x)$	Δ	$D(x)$	x	$N_\sigma(x)$	Δ	$D(x)$
100000	13863	13863	0.13863	6000000	936244	158572	0.156041
200000	28572	14712	0.14286	7000000	1095710	159466	0.15653
300000	43515	14940	0.14505	8000000	1255016	159306	0.156877
400000	58459	14944	0.146148	9000000	1414783	159767	0.157198
500000	73565	15106	0.14713	10000000	1574973	160190	0.157497
600000	88828	15263	0.148047	20000000	3184111	1609138	0.159206
700000	104062	15234	0.14866	30000000	4804331	1620220	0.160144
800000	119302	15240	0.149128	40000000	6430224	1625893	0.160756
900000	134758	15456	0.149731	50000000	8060163	1629939	0.161203
1000000	150232	15474	0.150232	60000000	9694467	1634304	0.161574
2000000	305290	155058	0.152645	70000000	11330312	1635845	0.161862
3000000	462110	156820	0.154037	80000000	12967239	1636927	0.16209
4000000	619638	157528	0.15491	90000000	14606549	1639310	0.162295
5000000	777672	158034	0.15553	100000000	16246940	1640391	0.162469

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