RATIONAL LINEAR SPACES ON HYPERSURFACES OVER QUASI-ALGEBRAICALLY CLOSED FIELDS

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ABSTRACT. Let $k = \mathbb{F}_q(t)$ be the rational function field over $\mathbb{F}_q$ and $f(\mathbf{x}) \in k[x_1, \dots, x_s]$ be a form of degree d. For $l \in \mathbb{N}$, we establish that whenever $s > l + \sum_{w=1}^{d} w^2 \binom{d-w+l-1}{l-1}$, the projective hypersurface $f(\mathbf{x}) = 0$ contains a k-rational linear space of projective dimension l. We also show that if $s > 1 + d(d+1)(2d+1)/6$ then for any k-rational zero $\mathbf{a}$ of $f(\mathbf{x})$ there are infinitely many s-tuples $(\varpi_1, \dots, \varpi_s)$ of monic irreducible polynomials over k, with the ϖ_i not all equal, and $f(a_1\varpi_1, \dots, a_s\varpi_s) = 0$. We establish in fact more general results of the above type for systems of forms over C_i-fields.

1. Introduction

In 1957, Birch [2] proved that any system of odd-degree forms over the rational numbers $\mathbb{Q}$ possesses a solution set containing a $\mathbb{Q}$-rational linear space of projective dimension l provided that the system has sufficiently many variables in terms of the number of forms, the degrees of the forms, and l. Much work has been put into establishing bounds for the particular case of systems of cubic forms (see, for instance, [7, 8, 14, 20]). However, it was not until 1998 when Wooley [19] provided the first explicit bounds for the general problem. More recently, Dietmann [6] proved the following result for a single odd-degree form.

Theorem 1.1. *Let $f(x_1, \dots, x_s) \in \mathbb{Q}[x_1, \dots, x_s]$ be a non-singular form of odd degree d. Let $l \in \mathbb{N}$ and*

$$s \geqslant 2^{1+(5+2^{d-1})d} d! d^{2^d+1} (l+1)^{d(1+2^{d-1})}.$$

Then, there exists a projective l-dimensional $\mathbb{Q}$-rational linear space of solutions to the hypersurface $f(x_1, \dots, x_s) = 0$.

One anticipates that similar conclusions are available when $\mathbb{Q}$ is replaced by the rational function field $\mathbb{F}_q(t)$. In this paper, we not only show that such results may be obtained,

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but we are able to prove substantially sharper conclusions with relatively simple proofs. It is our hope that the quantitative results for $\mathbb{F}_q(t)$ may shed light on what is to be expected in the classical case of $\mathbb{Q}$. The following theorem is a direct consequence of Theorem 3.1, where a similar statement is given in the more general setting of C_i-fields.

Theorem 1.2. *Suppose that for $1 \leqslant j \leqslant r$, the form $f_j(\mathbf{x}) \in \mathbb{F}_q(t)[x_1, \dots, x_s]$ is of degree d_j. Then, provided that*

$$s > \begin{cases} l + \sum_{j=1}^r \sum_{w=1}^{d_j} w^2 \binom{d_j - w + l - 1}{l-1}, & \text{when } l > 0, \\ \sum_{j=1}^r d_j^2, & \text{when } l = 0, \end{cases}$$

the set of solutions of the system

$$f_j(\mathbf{x}) = 0 \quad (1 \leqslant j \leqslant r) \tag{1.1}$$

contains a k-rational linear space of projective dimension l.

By combining Theorem 1.2, the Green-Tao Theorem for $\mathbb{F}_q[t]$ due to Lê [10], and the argument from [3] due to Brüdern, Dietmann, Liu, and Wooley, one can prove the following result.

Theorem 1.3. *Suppose that for $1 \leqslant j \leqslant r$, the form $f_j(\mathbf{x}) \in \mathbb{F}_q(t)[x_1, \dots, x_s]$ is of degree d_j and that*

$$s > 1 + \sum_{j=1}^r \frac{d_j(d_j+1)(2d_j+1)}{6}. \tag{1.2}$$

Then for any solution $\mathbf{a} = (a_1, \dots, a_s) \in \mathbb{F}_q(t)^s$ of (1.1) there exist infinitely many s-tuples $(\varpi_1, \dots, \varpi_s)$ of monic irreducible polynomials in $\mathbb{F}_q(t)$ with $\varpi_1, \dots, \varpi_s$ not all equal, such that

$$f_j(a_1\varpi_1, a_2\varpi_2, \dots, a_s\varpi_s) = 0, \quad 1 \leqslant j \leqslant r. \tag{1.3}$$

Remark 1.4. Under assumption (1.2), the hypothesis of Theorem 1.2 is satisfied with $l = 1$ and thus the system (1.1) is guaranteed to have a projective line of solutions.

Remark 1.5. The theorem is not true for arbitrary vectors $\mathbf{a}$. For example, let $\pi_1, \dots, \pi_s$ be distinct monic irreducible polynomials, $P = \pi_1 \cdots \pi_s$, $P_i = (P/\pi_i)^{d+1}$ $(1 \leqslant i \leqslant s)$, and $f(\mathbf{x}) = P_1 x_1^d + \cdots + P_s x_s^d$. Any solution $\mathbf{x} \in \mathbb{F}_q[t]^s$ to the equation $f(\mathbf{x}) = 0$ must satisfy $\pi_l^2 | x_l$ for $1 \leqslant l \leqslant s$. Thus (1.3) has no solution if $\mathbf{a}$ is a vector of constants from $\mathbb{F}_q$.

Remark 1.6. The conclusion of the theorem is trivially true for any solution $\mathbf{a}$ of (1.1) having some coordinate equal to zero. In this case one can simply let the ϖ_u for this coordinate position be arbitrary and set the remaining ϖ_u equal to each other. However, one can prove the following variation of Theorem 1.3 that avoids such trivial solutions.

Theorem 1.7. *Given any projective line of solutions of a homogeneous system of equations (1.1) in any number of variables, there exists a non-trivial point $\mathbf{a}$ on this line such that there are infinitely many s-tuples of monic irreducible polynomials $(\varpi_1, \dots, \varpi_s)$ satisfying (1.3) with the property that in the coordinate positions where $a_u \neq 0$, not all of the ϖ_u are equal.*

The proofs of Theorem 1.3 and Theorem 1.7 are given in Section 4.

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2. Quasi-algebraically closed fields

We begin by introducing some notation. Let k be a field. We say that a zero of a polynomial in several variables is *non-trivial* when it has a non-zero coordinate. We refer to a homogeneous polynomial as a *form*, and we call a polynomial having zero constant term a *Chevalley polynomial*. The set of zeros of a form (or system of forms) may be regarded as either a subset of affine space k^s or projective space $\mathbb{P}^{s-1}(k)$. Moreover any linear subspace of k^s of dimension m corresponds to a *linear subspace* of $\mathbb{P}^{s-1}(k)$ of dimension $m-1$. Associated to k is the polynomial ring $k[t]$ and the field of fractions $k(t)$. We recall that a field k is called *quasi-algebraically closed* if every non-constant form over k, having a number of variables exceeding its degree, possesses a non-trivial zero. In this context we recall the language of Lang [9]. We say that k is a *C_i-field* when any form of positive degree d lying in $k[\mathbf{x}]$, having more than d^i variables, necessarily possesses a non-trivial k-rational zero. Thus, quasi-algebraically closed fields are C_1-fields. We say that k is a *strongly C_i-field* when any Chevalley polynomial of positive degree d lying in $k[\mathbf{x}]$, having more than d^i variables, necessarily possesses a non-trivial k-rational zero. In this terminology, algebraically closed fields such as $\mathbb{C}$ are strongly C_0-fields, and from the Chevalley-Warning theorem (see [4] and [18]) it follows that the finite field $\mathbb{F}_q$ in q elements is a strongly C_1-field. Work of Lang [9] and Nagata [15], moreover, shows that algebraic extensions of (strongly) C_i-fields are (strongly) C_i, and that a transcendental extension, of transcendence degree j, over a (strongly) C_i-field is (strongly) C_{i+j}. In particular, $\mathbb{F}_q(t)$ is a strongly C_2-field.

We say that a form $\Psi(\mathbf{x}) \in k[x_1, \ldots, x_s]$ is *normic* when it satisfies the property that the equation $\Psi(\mathbf{x}) = 0$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. When such is the case, and the form $\Psi(\mathbf{x})$ has degree d and contains d^i variables, then we say that Ψ is *normic of order i*. Plainly, when k is a C_i-field, any normic form $\Psi(\mathbf{x})$ of degree d can have at most d^i variables. We note also that when $k = \mathbb{F}_q$, then for each natural number d there exist normic forms of degree d in d variables, and therefore of order 1. In order to exhibit such a form, consider a field extension L of $\mathbb{F}_q$ of degree d, and examine the norm form $\Psi(\mathbf{x})$ defined by considering the norm map from L to $\mathbb{F}_q$ with respect to a coordinate basis for the field extension of L over $\mathbb{F}_q$. Similarly, there are normic forms of order 2 over $\mathbb{F}_q(t)$ of each positive degree. Namely, if $\Psi(\mathbf{x}) : \mathbb{F}_q^d \to \mathbb{F}_q$ is a normic form of order 1 and degree d, by extending the domain of Ψ to $\mathbb{F}_q(t)$ and considering the form $\widetilde{\Psi} : \mathbb{F}_q(t)^{d^2} \to \mathbb{F}_q(t)$

defined by $\widetilde{\Psi}(\mathbf{x}) = \sum_{j=0}^{d-1} \Psi(x_{jd+1}, \ldots, x_{jd+d})t^j$, one obtains a normic form of order 2 and degree d.

We recall two theorems on C_i-theory relevant to our subsequent arguments.

Theorem 2.1. *Let k be a C_i-field, and suppose that for $1 \leqslant j \leqslant r$, the form $g_j(\mathbf{x}) \in k[x_1, \ldots, x_s]$ is of degree d_j. Suppose also that there are normic forms over k of order i of each positive degree. Then whenever $s > \sum_{j=1}^{r} d_j^i$, the system of equations $g_j(\mathbf{x}) = 0$ $(1 \leqslant j \leqslant r)$ possesses a non-trivial k-rational solution.*

Proof. This is Theorem 4 of Lang [9]. □

Theorem 2.2. *Let k be a strongly C_i-field, and suppose that for $1 \leqslant j \leqslant r$, the Chevalley polynomial $g_j(\mathbf{x}) \in k[x_1, \ldots, x_s]$ is of degree at most d. Suppose also that $s > rd^i$. Then the system of equations $g_j(\mathbf{x}) = 0$ $(1 \leqslant j \leqslant r)$ possesses a non-trivial k-rational solution.*

Proof. This is Theorem 1b of Nagata [15]. □

3. Finding linear spaces of solutions via C_i-theory

Theorem 3.1. *Let k be a C_i-field, and suppose that for $1 \leqslant j \leqslant r$, the form $f_j(\mathbf{x}) \in k[x_1, \ldots, x_s]$ is of degree d_j. Suppose also that there are normic forms over k of order i of each positive degree. Then, provided that*

$$s > \begin{cases} l + \sum_{j=1}^{r} \sum_{w=1}^{d_j} w^i \binom{d_j - w + l - 1}{l-1}, & \text{when } l > 0, \\ \sum_{j=1}^{r} d_j^i, & \text{when } l = 0, \end{cases} \tag{3.1}$$

the system of equations $f_j(\mathbf{x}) = 0\,(1 \leqslant j \leqslant r)$ possesses a solution set containing a k-rational linear space of projective dimension l.

Since $\mathbb{F}_q(t)$ is a C_2-field with normic forms of order 2 for each positive degree, we immediately deduce Theorem 1.2. Theorem 3.1 follows readily from the work of Leep and Schmidt [13, Equation (3.1)] and Theorem 2.1. For the convenience of the reader, we give a proof here that follows the same line of argument as in [13].

Proof. We prove the theorem by induction on l. When $l = 0$, the theorem is equivalent to Theorem 2.1. Assume that $m \in \mathbb{N}$ and that the theorem holds when $l = m - 1$. We now establish that the theorem holds when $l = m$. Suppose that

$$s > m + \sum_{j=1}^{r} \sum_{w=1}^{d_j} w^i \binom{d_j - w + m - 1}{m - 1}.$$

By noting that the right-hand-side of (3.1) is an increasing function of l, we obtain from the induction assumption that the system $f_j(\mathbf{x}) = 0$ $(1 \leqslant j \leqslant r)$ contains a k-rational linear space of solutions in affine space of dimension m. By applying a linear change of variables, we may assume that $\mathbf{e}_1 = (1, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_m$ is a basis for this subspace, that is, the forms f_j are identically zero on k^m when we set $x_{m+1} = x_{m+2} = \cdots = x_s = 0$. Thus, for $1 \leqslant j \leqslant r$ we can write

$$f_j(\mathbf{x}) = \sum_{\substack{b_1,\dots,b_m \in \mathbb{Z}_{\geqslant 0} \\ b_1+\cdots+b_m < d_j}} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m} f_{j;\mathbf{b}}(x_{m+1}, \dots, x_s) + g_j(x_1, \dots, x_m) \tag{3.2}$$

where each $f_{j;\mathbf{b}}(x_{m+1}, \dots, x_s)$ is a form of degree $d_j - b_1 - b_2 - \cdots - b_m > 0$ in $s - m$ variables, and $g_j(x_1, \dots, x_m)$ is a form of degree d_j that is identically zero on k^m (although not necessarily the zero polynomial).

Note that for $1 \leqslant j \leqslant r$, by [16, Theorem 2.3], there are $\binom{d_j-w+m-1}{m-1}$ choices of $(b_1, \dots, b_m) \in (\mathbb{Z}_{\geqslant 0})^m$ with $b_1 + \cdots + b_m = d_j - w$, which would make $f_{j;\mathbf{b}}(x_{m+1}, \dots, x_s)$ a degree-w form. By Theorem 2.1, it follows that if

$$s - m > \sum_{j=1}^{r} \sum_{w=1}^{d_j} w^i \binom{d_j - w + m - 1}{m - 1}, \tag{3.3}$$

we can find a non-trivial solution $(x_{m+1}, \dots, x_s)$ to the system

$$f_{j;\mathbf{b}}(x_{m+1}, \dots, x_s) = 0 \qquad (1 \leqslant j \leqslant r,\ b_1 + \cdots + b_m < d_j).$$

Then, upon recalling (3.2) and the fact that g_j is identically zero, we see that

$$\{(0, \dots, 0, x_{m+1}, \dots, x_s), \mathbf{e}_1, \dots, \mathbf{e}_m\}$$

is a basis of a projective m-dimensional k-rational linear space of solutions for the system of forms $f_j(\mathbf{x}) = 0\,(1 \leqslant j \leqslant r)$. This completes the proof of the theorem. □

Remark 3.2. For algebraically closed fields (C_0-fields) the inequality in Theorem 3.1 simplifies to

$$s > l + \sum_{j=1}^{r} \binom{d_j + l - 1}{l}, \tag{3.4}$$

for $l \geqslant 0$, noting in the case $l = 0$ that $\binom{n}{0} = 1$ for any integer n. In the case $l = 0$ the estimate is optimal, but for $l > 0$ we do not know how sharp it is in general, and suspect that one should be able to do better. For $r = 1$ and $d = 2$, inequality (3.4) reads $s > 2l + 1$, and is optimal. Indeed, any non-degenerate quadratic form in $s = 2l + 1$ variables can only vanish on a linear subspace of projective dimension at most $l - 1$.

In certain cases, improvements are available. If $r = 2$, $d_1 = d_2 = 2$ and $l \geqslant 1$ then inequality (3.4) reads $s > 3l + 2$, but in this case it follows from a result of Amer [1, Satz 8] (for fields of characteristic $\neq 2$) and Leep [12] (for any field), and the fact that for algebraically closed fields K, $K(t)$ is a C_1-field, that it suffices to take $s > 2l + 2$.

Moreover, this bound is optimal. If $r = 1$, $d = 3$ and $l = 1$, inequality (3.4) reads $s > 4$, but in this case it is well known that $s = 4$ suffices [17, Section 1.6, Theorem 10].

Remark 3.3. For C_1-fields, such as finite fields, the inequality of Theorem 3.1 simplifies to

$$s > l + \sum_{j=1}^{r} \binom{d_j + l}{l + 1}, \tag{3.5}$$

for $l \geqslant 0$. For $l = 0$, this bound is sharp, but for $l > 0$ we expect improvements to be available in general. For systems of quadratic forms over $\mathbb{F}_q$, the bound in (3.5) was given earlier by Leep [11, Corollary 2.4(ii)] and by the first author [5, Lemma 3(a)]. For a pair of quadratic forms, it states that if $s \geqslant 3l + 5$ then the system vanishes on a linear subspace of projective dimension l. However, in this case, it is known [5, Lemma 3(c)] that one only needs $s \geqslant 2l + 5$.

For the case of a single form of degree d, the bound in (3.5) reads

$$s > l + \binom{d + l}{l + 1} = l + \binom{l + d}{d - 1} = \frac{1}{(d-1)!} l^{d-1} + O_d(l^{d-2}),$$

viewing the latter as a polynomial in l with d fixed. It was observed in [5, Section 6] that for $d < q$ and $s < \frac{(l+1)^{d-1}}{d!}$ there exists a form of degree d over $\mathbb{F}_q$ in s variables not vanishing on any linear subspace of projective dimension l. For the case of a cubic form this was refined slightly by Dietmann [7, Lemma 6]. Thus, as a polynomial in l, the optimal bound in this case is somewhere between $\frac{l^{d-1}}{d!}$ and $\frac{l^{d-1}}{(d-1)!}$ in the leading term. It would be nice to pin down the discrepancy between these two values.

Remark 3.4. A closer examination of the proof of Theorem 3.1 reveals a slightly stronger conclusion when $l > 0$. Under the hypotheses of the theorem, if $\mathbf{a}$ is a given non-trivial solution of the system $f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0$ then in fact we obtain a k-rational linear space of solutions of projective dimension l containing $\mathbf{a}$. Indeed, the constructive nature of the proof of the theorem shows that given an $(m-1)$-dimensional subspace of solutions, there exists an m-dimensional subspace containing the given subspace provided that the number of variables is of the requisite size.

If we drop the hypothesis on k having normic forms of order i for each positive degree and add the requirement that k is a strongly C_i-field, we obtain the slightly weaker result of the next theorem.

Theorem 3.5. *Let k be a strongly C_i-field, and suppose that for $1 \leqslant j \leqslant r$, the form $f_j(\mathbf{x}) \in k[x_1, \ldots, x_s]$ is of degree d_j. Let $d = \max_{1 \leqslant j \leqslant r} d_j$. Suppose that*

$$s > l + d^i \sum_{j=1}^{r} \binom{d_j + l - 1}{l}.$$

Then, the system of equations $f_j(\mathbf{x}) = 0 \, (1 \leqslant j \leqslant r)$ possesses a solution set containing a k-rational linear space of projective dimension l.

Proof. We mimic the proof of Theorem 3.1, applying Theorem 2.2 instead of Theorem 2.1. The inequality in (3.3) is replaced with

$$s - m > d^i \sum_{j=1}^{r} \sum_{w=1}^{d_j} \binom{d_j - w + m - 1}{m-1}.$$

The sum over w can be written as $\displaystyle\sum_{\alpha=m-1}^{m+d_j-2} \binom{\alpha}{m-1}$, and the theorem follows upon applying the combinatorial identity [16, Section 2.8, Exercise 5]

$$\sum_{\alpha=\beta}^{\gamma} \binom{\alpha}{\beta} = \binom{\gamma+1}{\beta+1}.$$ □

In particular, using $\binom{d+l-1}{l} \leqslant d^l$, we see that it suffices to take $s > l + rd^{i+l}$ in order to obtain a k-rational linear space of projective dimension l.

4. Proofs of Theorem 1.3 and Theorem 1.7

Through the use of an argument due to Brüdern, Dietmann, Liu, and Wooley [3], we will apply Theorem 1.2 and the Green-Tao Theorem for $\mathbb{F}_q[t]$ due to Lê [10] to obtain Theorem 1.3. We recall that for a subset $\mathcal{A}$ of the set of irreducible polynomials $\mathcal{P}$ in $\mathbb{F}_q[t]$, the relative upper density, $\overline{d_{\mathcal{P}}}(\mathcal{A})$, of $\mathcal{A}$ in $\mathcal{P}$ is defined by

$$\overline{d}_{\mathcal{P}}(\mathcal{A}) = \varlimsup_{N\to\infty} \frac{\#\{f \in \mathcal{A} : \deg(f) < N\}}{\#\{f \in \mathcal{P} : \deg(f) < N\}}.$$

Theorem 4.1. [10, Theorem 2] *For any $k > 0$, there exist polynomials $f, g \in \mathbb{F}_q[t]$, $g \neq 0$, such that the polynomials $f + Pg$, where P runs over all polynomials in $\mathbb{F}_q[t]$ of degree less than k, are all irreducible. Furthermore, such configurations can be found in any set of positive relative upper density among the irreducible polynomials.*

Remark 4.2. In particular, as Lê notes, the set of monic irreducible polynomials has positive upper density in $\mathcal{P}$, and so we conclude that there exist f, g, $g \neq 0$, such that $f + Pg$ is a monic irreducible polynomial for all P of degree less than a given k. Moreover, by repeated applications of the theorem, one can in fact obtain infinitely many pairs (f, g) satisfying the conclusion of the theorem.

Proof of Theorem 1.3. Suppose that for $1 \leqslant j \leqslant r$, the form $f_j(\mathbf{x}) \in \mathbb{F}_q(t)[x_1, \dots, x_s]$ is of degree d_j. By Theorem 1.2 with $l = 1$ and Remark 3.4, given any non-trivial solution $\mathbf{a} \in \mathbb{F}_q(t)^s$ of the system

$$f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0 \tag{4.1}$$

there exists a projective $\mathbb{F}_q(t)$-rational line of solutions of (4.1) containing $\mathbf{a}$, provided that

$$s > 1 + \sum_{j=1}^{r} \sum_{w=1}^{d_j} w^2 = 1 + \sum_{j=1}^{r} \frac{d_j(d_j+1)(2d_j+1)}{6}.$$

By homogeneity, we may assume that $\mathbf{a} \in \mathbb{F}_q[t]^s$, and there exists a vector $\mathbf{b} \in \mathbb{F}_q[t]^s$, with $\mathbf{a}$ and $\mathbf{b}$ linearly independent over $\mathbb{F}_q(t)$, satisfying $f_j(\alpha\mathbf{a} + \beta\mathbf{b}) = 0$ for all $1 \leqslant j \leqslant r$ and all $\alpha, \beta \in \mathbb{F}_q[t]$.

If some coordinate of $\mathbf{a}$ is zero, then the theorem follows trivially. Indeed, say $a_1 = 0$, let ϖ_1, ϖ_2 be any two distinct monic irreducible polynomials in $\mathbb{F}_q[t]$ and set $\varpi_3 = \varpi_4 = \cdots = \varpi_s = \varpi_2$. Then

$$(a_1\varpi_1, a_2\varpi_2, \ldots, a_s\varpi_s) = (0, a_2\varpi_2, \ldots, a_s\varpi_2) = \varpi_2(a_1, a_2, \ldots, a_s)$$

is a solution of (4.1) since $\mathbf{a}$ is a solution.

Next, suppose that $a_u \neq 0$ for $1 \leqslant u \leqslant s$. Let $\tilde{a} = a_1 \ldots a_s$, and put $\tilde{a}_u = \tilde{a}/a_u$ for $1 \leqslant u \leqslant s$. Set $M = \max_{1 \leqslant u \leqslant s} \deg(\tilde{a}_u b_u)$. By Remark 4.2, there exist infinitely many pairs of nonzero polynomials $y, z \in \mathbb{F}_q[t]$ for which the set $\{y + zw : \deg(w) \leqslant M\}$ consists entirely of monic irreducible polynomials. For any such pair (y, z), define $\varpi_u = y + z\tilde{a}_u b_u$ for $1 \leqslant u \leqslant s$. Since $\deg(\tilde{a}_u b_u) \leqslant M$ we have that ϖ_u is a monic irreducible polynomial for $1 \leqslant u \leqslant s$. Also, $a_u\varpi_u = ya_u + (z\tilde{a})b_u$ for $1 \leqslant u \leqslant s$. Therefore, the vector

$$(a_1\varpi_1, \ldots, a_s\varpi_s) = y\mathbf{a} + (z\tilde{a})\mathbf{b} \tag{4.2}$$

is on our given projective line of solutions of (4.1). □

Proof of Theorem 1.7. Suppose that we are given a projective line of solutions of the homogeneous system (4.1). Then there is a pair of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q[t]^s$ on this line, linearly independent over $\mathbb{F}_q(t)$, satisfying $f_j(\alpha\mathbf{a} + \beta\mathbf{b}) = 0$ for all $1 \leqslant j \leqslant r$ and all $\alpha, \beta \in \mathbb{F}_q[t]$. Without loss of generality, we may assume that at least one of a_u, b_u is nonzero for $1 \leqslant u \leqslant \eta$ and that $a_u = b_u = 0$ for $\eta < u \leqslant s$. Note that η satisfies $2 \leqslant \eta \leqslant s$ because $\mathbf{a}$ and $\mathbf{b}$ are linearly independent over $\mathbb{F}_q(t)$. Furthermore, since $\mathbb{F}_q[t]$ is infinite, there exists a $\lambda \in \mathbb{F}_q[t]$ such that $\mathbf{a} + \lambda\mathbf{b}$ has all nonzero coordinates in the first η places, and so replacing $\mathbf{a}$ with this vector we may assume that $a_u \neq 0$ for $1 \leqslant u \leqslant \eta$ and that $a_u = b_u = 0$ for $\eta < u \leqslant s$.

Let $\tilde{a} = a_1 \ldots a_\eta$, and put $\tilde{a}_u = \tilde{a}/a_u$ for $1 \leqslant u \leqslant \eta$. Set $M = \max_{1 \leqslant u \leqslant \eta} \deg(\tilde{a}_u b_u)$. By Remark 4.2, there exist infinitely many pairs of nonzero polynomials $y, z \in \mathbb{F}_q[t]$ for which the set $\{y + zw : \deg(w) \leqslant M\}$ consists entirely of monic irreducible polynomials. For any such pair (y, z), define

$$\varpi_u = \begin{cases} y + z\tilde{a}_u b_u, & \text{when } 1 \leqslant u \leqslant \eta, \\ \text{any monic irreducible polynomial}, & \text{when } \eta < u \leqslant s. \end{cases}$$

Since $\deg(\tilde{a}_u b_u) \leqslant M$ for $1 \leqslant u \leqslant \eta$, we have that ϖ_u is a monic irreducible polynomial for $1 \leqslant u \leqslant \eta$, and thus, ϖ_u is a monic irreducible polynomial for $1 \leqslant u \leqslant s$. Also, for $1 \leqslant u \leqslant \eta$ we have $a_u \varpi_u = a_u(y + z\tilde{a}_u b_u) = ya_u + (z\tilde{a})b_u$, while for $\eta < u \leqslant s$ we trivially have $a_u \varpi_u = 0 = ya_u + (z\tilde{a})b_u$, since $a_u = b_u = 0$. Thus, we again have (4.2) and see that $(\varpi_1 a_1, \ldots, \varpi_s a_s)$ is a solution of (4.1). Note that because $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, the polynomials $\tilde{a}_u b_u = \tilde{a} b_u / a_u$ $(1 \leqslant u \leqslant \eta)$ cannot all be the same, implying that our monic irreducible polynomials $\varpi_1, \ldots, \varpi_\eta$ cannot all be equal. □

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