Rational linear spaces on hypersurfaces over quasi-algebraically closed fields

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Outline

1. Overview
    - Introduction
    - What is C_i theory?
2. Main result
    - Statement of the main result ($C_i^\sharp$)
    - Sketch of the proof
3. Further result
    - Re-visit to $\mathbb{F}_q(t)$ and $\mathbb{Q}$
    - One advanced application
4. References
5. Acknowledgement

Notation

- $\mathbb{F}_q$: the finite field of characteristic p with $q := p^n$ elements, for some prime p and $n \in \mathbb{N}$
- $\mathbb{F}_q(t)$: the field of quotients of $\mathbb{F}_q[t]$, the polynomial ring over $\mathbb{F}_q$. $\mathbb{F}_q(t)$ is called a *function field*.

Some definitions & a question

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Let k be a field. Then a form over k is a homogeneous polynomial with coefficients in k. For instance, $x^3 + y^2z - 7xyz$ is homogeneous, while $y^2 - x + 1$ is not.

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A zero of a form $f(x_1, x_2, \ldots, x_s)$ is called non-trivial if any of $x_1, x_2, \ldots, x_s$ is non-zero.

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Consider forms over $\mathbb{Q}$. Under what conditions do forms over $\mathbb{Q}$ have a projective l-dimensional linear space of simultaneous zeroes?

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Answer: sufficiently many variables

Theorem (R. Dietmann, 2010)

Let $F(X_1, X_2, \ldots, X_s) \in \mathbb{Q}[X_1, X_2, \ldots, X_s]$ be a non-singular form of odd degree d, let $l \in \mathbb{N}$ and let

$$s \geq 2^{1+(5+2^{d-1})d}\, d!\, d^{2^d+1}\, (l+1)^{d(1+2^{d-1})}.$$

Then there exists a projective l-dimensional rational linear space $V \subset \mathbb{Q}^s$ such that $F(\mathbf{x}) = 0$ for all $x \in V$.

For quintic forms (w/o the non-singularity condition...)

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ONE WIDELY-BELIEVED HEURISTIC....

In number theory and algebraic geometry, one often finds that function fields and the rationals have very similar arithmetics.

$$\mathbb{F}_q(t) \Longleftrightarrow \mathbb{Q}$$

Our results show that, in $\mathbb{F}_q(t)$, the lower bound on the number of variables is considerably better. While it is hard to prove such a bound in $\mathbb{Q}$, our results nonetheless provide insight on what the actual bound on $\mathbb{Q}$ is likely to be.

Our result over $\mathbb{F}_q(t)$

The following is the special case of our main result:

Theorem

Let $f_1, \ldots, f_r \in \mathbb{F}_q(t)[x_1, \ldots, x_s]$ be forms of degrees $d_1, \ldots, d_r$, respectively. Then provided that

$$s > \begin{cases} l + \sum_{j=1}^{r} \sum_{w=1}^{d_j} w^2 \binom{d_j - w + l - 1}{l - 1} & \text{if } l \geq 1, \\ \sum_{j=1}^{r} d_j^2 & \text{if } l = 0, \end{cases}$$

there exists a projective l-dimensional $\mathbb{F}_q(t)$-linear space of simultaneous zeroes.

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Definitions of C_i and $C_i^\sharp$ fields

Definition

A field k is called C_i if any form of degree d with coefficients in k having more than d^i variables has a non-trivial zero. Therefore, any algebraically closed field is C_0. If k is C_1, then k is quasi-algebraically closed.

Definition

Let k be a C_i field, and let $N(\mathbf{x})$ be a form of degree d with d^i variables with coefficients in k. If $N(\mathbf{x})$ only has a trivial zero, then $N(\mathbf{x})$ is called a normic form of order i.

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A field k is called $C_i^\sharp$ if k is a C_i field admitting normic forms of order i for every degree.

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Key ideas from C_i theory

Theorem (Lang–Nagata Theorem for C_i fields)

Let k be a C_i field, and suppose that for $1 \leq j \leq r$, the form $g_j(\mathbf{x}) \in k[x_1, \ldots, x_s]$ is of degree at most d. Suppose also that $s > rd^i$. Then the system of forms $g_j(\mathbf{x}) = 0\,(1 \leq j \leq r)$ possesses a non-trivial k-rational solution.

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Here is the (generalized) main result:

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4. Re-write each form in a “nice-looking" shape
5. Use combinatorics and apply C_i theory to finish the proof.

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- Now, suppose

$$s > m + \sum_{j=1}^{r} \sum_{w=1}^{d_j} w^i \binom{d_j - w + m - 1}{m - 1}, \qquad (2)$$

whose right-hand side is (obviously) larger than the RHS of (1).

ON LINEAR CHANGE OF VARIABLES

Therefore, we can assume that there exist m-linearly independent vectors generating a projective $(m-1)$-dimensional linear space of solutions. Via a linear change of variables, we can convert the basis to the standard basis, i.e.,$\{e_1, e_2, \ldots, e_m\}$. We will prove that as long as s satisfies (2), i.e.

$$s > m + \sum_{j=1}^{r} \sum_{w=1}^{d_j} w^i \binom{d_j - w + m - 1}{m-1},$$

then we can construct a *non-trivial* vector $(\underbrace{0, \ldots, 0}_{m \text{ times}}, y_{m+1}, \ldots, y_s)$.

Re-write each form "nicely"

Re-write each form $f_j(\mathbf{x})$ $(1 \leq j \leq r)$ in the following way:

$$f_j(\mathbf{x}) = \sum_{\substack{a_1,\dots,a_m \in \mathbb{Z} \cup \{0\} \\ 0 \leq a_1 + \dots + a_m < d_j}} x_1^{a_1} x_2^{a_2} \dots x_{m-1}^{a_{m-1}} x_m^{a_m} f_{j;\mathbf{a}}(x_{m+1}, \dots, x_s)$$

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- Note that if $a_1 + \dots + a_m = d_j$, then $f_{j;\mathbf{a}}$ is constant.
- Thus, even though we put a non-zero entry to where $\mathbf{0}$ is, we still have
$$\sum_{\substack{a_1,\dots,a_m \in \mathbb{Z} \cup \{0\} \\ a_1 + \dots + a_m = d_j}} b_1^{a_1} b_2^{a_2} \dots b_{m-1}^{a_{m-1}} b_m^{a_m} f_{j;\mathbf{a}}(\mathbf{c}) = 0.$$

Combinatorics and the application of C_i theory

We want to all the auxiliary forms to vanish. In order to use C_i theory, we need to count the number of auxiliary forms of some degree.

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- Answer: $\binom{d_j - w + m - 1}{m-1}$ ways.

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So, by the L–N Theorem for $C_i^\sharp$ fields, we have

$$s - m > \sum_{j=1}^{r} \sum_{w=1}^{d_j} w^i \binom{d_j - w + m - 1}{m - 1},$$

so the result follows. □

If not $C_i^\sharp$ but just C_i, then the final inequality becomes (by the L–N Theorem for C_i fields)

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by applying the combinatorial identity $\sum_{\alpha=\beta}^{\gamma} \binom{\alpha}{\beta} = \binom{\gamma+1}{\beta+1}$.

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Re-visit to our bound for $\mathbb{F}_q(t)$

Fact: $\mathbb{F}_q(t)$ is $C_2^\sharp$. Along with this fact and our main result, the following theorem from the introduction directly follows:

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Let $f_1, \ldots, f_r \in \mathbb{F}_q(t)[x_1, \ldots, x_s]$ be forms of degrees $d_1, \ldots, d_r$, respectively. Then provided that

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ADVANCED APPLICATION OF THE MAIN RESULT

Applying some fancy machinery (such as the Green–Tao theorem for $\mathbb{F}_q(t)$) and the theorem from the previous slide, one can prove the following:

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Suppose that, for $1 \le j \le r$, the form $f_j(\mathbf{x}) \in \mathbb{F}_q(t)[x_1, x_2, \dots, x_s]$ is of degree d_j. Then, provided that

$$s > 1 + \sum_{j=1}^{r} \frac{d_j(d_j+1)(2d_j+1)}{6},$$

there exists a fixed vector $(c_1, \dots, c_s) \in \mathbb{F}_q[t]^s$, with $\gcd(c_1, \dots, c_s) = 1$, such that the system of forms $f_j(c_1\varpi_1, c_2\varpi_2, \dots, c_s\varpi_s) = 0$ for $1 \le j \le r$ possesses infinitely many solutions in monic irreducible polynomials $\varpi_1, \dots, \varpi_s$, not all equal.

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Suppose that, for $1 \leq j \leq r$, the form $f_j(\mathbf{x}) \in \mathbb{F}_q(t)[x_1, x_2, \ldots, x_s]$ is of degree d_j. Then, provided that

$$s > 1 + \sum_{j=1}^{r} \frac{d_j(d_j+1)(2d_j+1)}{6},$$

there exists a fixed vector $(c_1, \ldots, c_s) \in \mathbb{F}_q[t]^s$, with $\gcd(c_1, \ldots, c_s) = 1$, such that the system of forms $f_j(c_1\varpi_1, c_2\varpi_2, \ldots, c_s\varpi_s) = 0$ for $1 \leq j \leq r$ possesses infinitely many solutions in monic irreducible polynomials $\varpi_1, \ldots, \varpi_s$, not all equal.

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